

Forced Oscillations in a Linear System

Manual

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Annotation. The manual includes a description of the simulated physical system and a summary of the relevant theoretical material for students as a prerequisite for the virtual lab “Forced Oscillations of Linear Torsion Pendulum.” The manual includes also a set of theoretical and experimental problems to be solved by students on their own, as well as various assignments which the instructor can offer students for possible individual mini-research projects.

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1 Summary of the Theory

This manual is concerned with forced oscillations of a torsion spring pendulum driven by an external *sinusoidal* force. In the model of the physical system adopted here, a *kinematic mode* of excitation is used: one part of the system (a driving rod) is constrained to execute simple harmonic motion. We consider both steady-state forced oscillations and transient processes, the latter depending on the initial conditions. The decomposition of a transient process into the sum of sinusoidal oscillations with the frequency of the external force, and damped free oscillations with the natural frequency, is examined. A preliminary study of free oscillations (lab “Free Oscillations of a Linear Torsion Pendulum”) is strongly recommended.

1.1 General Concepts

In the conventional classification of oscillations by their mode of excitation, oscillations are called *forced* if an oscillator is subjected to an external periodic influence whose effect on the system can be expressed by a separate term, a periodic function of the time, in the differential equation of motion. We are interested in the response of the system to the periodic external force.

The behavior of oscillatory systems under periodic external forces is one of the most important topics in the theory of oscillations. A noteworthy distinctive characteristic of forced oscillations is the phenomenon of *resonance*, in which a small periodic disturbing force can produce an extraordinarily large response in the oscillator. Resonance is found everywhere in physics and so a basic understanding of this fundamental problem has wide and various applications.

The phenomenon of resonance depends upon the whole functional form of the driving force and occurs over an extended interval of time rather than at some particular instant.

In the case of unforced (free, or natural) oscillations of an isolated system, motion is initiated by an external influence acting *before* a particular instant. This influence determines the mechanical state of the system, that is, the displacement and the velocity of the oscillator, at the *initial instant*. These in turn determine the amplitude and phase of subsequent free oscillations. Frequency and damping of such oscillations are determined by the physical properties of the system.

On the other hand, the characteristics of forced oscillations generated by a periodic external influence depend not only on the initial conditions and physical properties of the oscillator but also on the nature of the external disturbance, that is, on its amplitude and (primarily) on frequency.

1.2 Discussion of the Physical System

To study forced oscillations in a linear system excited by a sinusoidal external force, we consider here the same torsion spring pendulum used in the lab devoted to free oscillations, namely, a balanced flywheel attached to one end of a spiral spring. The flywheel turns about its axis of rotation under the restoring torque of the spring, much like the devices used in mechanical watches. However, unlike the situation of free oscillations in which the other end of the spring is fixed, now this end is attached to an exciter, which is a rod that can be turned back and forth about an axis common with the axis of rotation of the flywheel. A schematic diagram of the driven torsion oscillator as displayed on the computer screen is shown in figure 1.

A mechanical system such as this one is ideal for the study of resonance because it is possible to see directly what is happening. When the driving rod (the exciter) is turned through a given angle, the equilibrium position of the flywheel is displaced through the same angle, alongside the rod. The flywheel can execute free damped oscillations about this displaced position. For weak and for moderate friction the angular frequency of these oscillations is close to the natural frequency ω_0 of the flywheel.

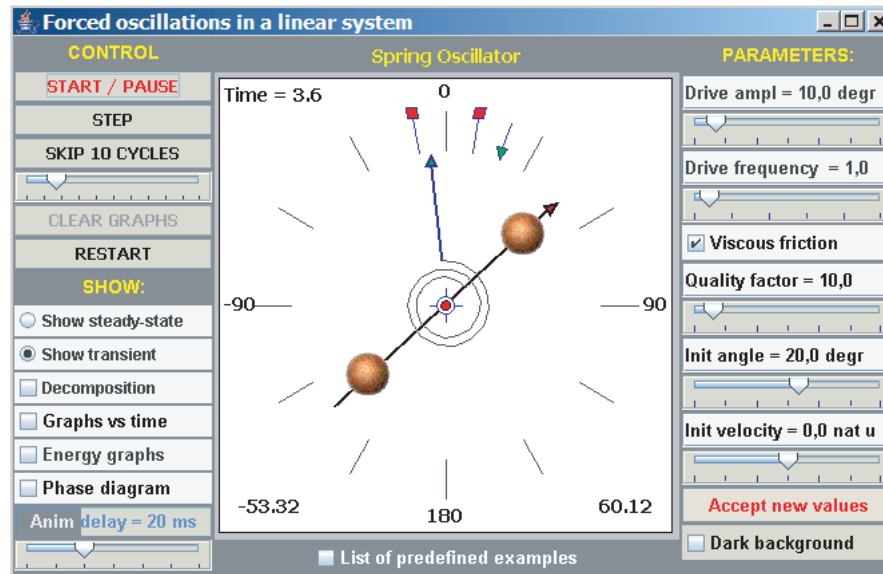


Figure 1: The torsion spring oscillator excited by a given sinusoidal motion of the driving rod attached to the spiral spring.

This frequency depends on the torsion spring constant D and the moment of inertia J of the flywheel: $\omega_0 = \sqrt{D/J}$.

If the rod is forced to execute a periodic oscillatory motion, the flywheel is subjected to the action of a periodic external torque. This action is an example of the *kinematic excitation* of forced oscillations. This method of excitation is characterized by a given periodic motion of some part of the system. The kinematic mode of excitation is chosen here for the computer simulations of forced oscillations because the motion of the exciting rod can be displayed directly on the computer screen. Computer experiments with the system can show clearly, among other things, the phase shift between the exciter and the flywheel, and the ratio of their amplitudes.

Another possible mode of excitation of forced oscillations is characterized by a given periodic external force whose value does not depend on the position and velocity of the excited oscillator. This mode of excitation is called *dynamic*. Such excitation is difficult to display on the screen because it does not arise from the mechanical *motion* of the external source. Moreover, this mode of excitation of a mechanical system is not easy to realize experimentally. Nevertheless, in most textbooks forced oscillations are treated under the assumption that an oscillatory system is excited by a given periodic force.

The differential equations describing forced oscillations are the same for both modes of excitation. The physical differences appear primarily in the character of energy transformations. When the excitation is kinematic, the equilibrium position of the flywheel moves alongside the moving rod. The corresponding parabolic potential well then also moves as a whole to and fro alongside the rod. On the other hand, when the excitation is dynamic, the potential well is stationary. We discuss these differences below.

1.3 The Differential Equation for Forced Oscillations

We let the exciting rod be constrained by some external source to execute simple harmonic motion about a middle position (the vertical in Fig. 4.1). The amplitude of the rod is ϕ_0 and the angular frequency is ω . The angular displacement of the rod $\phi(t)$ varies with time t sinusoidally:

$$\phi(t) = \phi_0 \sin \omega t. \quad (1)$$

If at some instant t the flywheel is displaced through an angle $\varphi(t)$ from the central position (which is the origin of the scale shown in figure 1), and the rod is simultaneously displaced through an angle ϕ , the spring exerts a torque $-D(\varphi - \phi) = -D\varphi + D\phi_0 \sin \omega t$ on the flywheel because the spring is strained through the angle $\varphi - \phi$. (Compare this torque with the torque $-D\varphi$ for the case of free oscillations.) Hence, in the absence of friction, the differential equation of rotation of the flywheel with the moment of inertia J is:

$$J\ddot{\varphi} = -D\varphi + D\phi_0 \sin \omega t. \quad (2)$$

This equation is also the differential equation of forced oscillations excited by a given external sinusoidally varying torque $D\phi_0 \sin \omega t$ whose constant amplitude is $D\phi_0$. That is, the equation of motion is the same for forced oscillations excited by these two modes (an oscillating motion of the rod and a sinusoidal external torque).

Dividing both sides of this equation by the moment of inertia J of the flywheel and introducing the notation $\omega_0 = \sqrt{D/J}$ for the natural frequency ($\omega_0^2 = D/J$), we rewrite the above equation in a canonical form:

$$\ddot{\varphi} + \omega_0^2 \varphi = \omega_0^2 \phi_0 \sin \omega t. \quad (3)$$

In the presence of viscous friction whose torque is proportional to the angular velocity $\dot{\varphi}$ of the flywheel, we must add the appropriate frictional term to the differential equation of motion describing forced oscillations:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2 \varphi = \omega_0^2 \phi_0 \sin \omega t. \quad (4)$$

The damping constant γ characterizes the strength of viscous friction. As in the case of free oscillations, it is related to the dimensionless quality factor Q by the expression $2\gamma/\omega_0 = 1/Q$.

The position of the modelled physical system at any instant is determined by two angular coordinates, namely, by the angles φ and ϕ . But the coordinate ϕ is entirely determined by external conditions. (The physical meaning of ϕ is the instantaneous position of the rod which executes a given motion.) The angle ϕ it is not a “free” coordinate, and so the system does not actually have a second degree of freedom. The only “free” coordinate (that is, the coordinate whose functional dependence on time is yet to be determined) is the angle φ , which gives the deflection of the flywheel from the central position. To find this unknown function $\varphi(t)$, we need only one differential equation, Eq. (4). The differential equation corresponding to the second coordinate ϕ can be used to find the external torque which must be exerted on the rod in order to provide its given sinusoidal motion. The external source of such torque inevitably experiences the reaction of the oscillator.

1.4 The Principle of Superposition

Our investigation of forced oscillations in a system whose differential equation of motion is *linear* is facilitated by the *principle of superposition*. This principle states that if several external forces act simultaneously on a linear system, the forced oscillations caused by each force acting separately are to be added together (superimposed) to get the complete solution. In other words, in linear systems there is no interaction (no mutual influence) of the individual oscillations excited by several external forces acting simultaneously.

It follows from the principle of superposition that, in addition to the forced oscillations caused by a given external force, a linear oscillator can simultaneously execute free damped oscillations. These free (or natural) oscillations may be thought of as arising from a null external force on the right-hand side of the differential equation of forced oscillations, Eq. (4). We can always infer that along with a given driving force a null force is also present. These natural oscillations are excited when an external force is switched on (or when its amplitude or initial phase is changed). A superposition of such damped natural oscillations and driven forced oscillations of constant amplitude occurs during a *transient process*, when forced oscillations, over a period of time, acquire the frequency of the external force and a constant amplitude. The duration τ of this transient process of establishing forced *steady-state oscillations* equals (in the general case) the duration of damping of free oscillations: $\tau = 1/\gamma$.

1.5 Steady-state Forced Oscillations

During some time after the external force has been activated (after the rod has begun its given periodic motion), the transient natural oscillations inevitably damp out. Since only these oscillations depend on the initial conditions, we can say figuratively that the oscillator eventually “forgets” its initial state, and its forced oscillations become steady: the flywheel executes harmonic oscillations of a constant amplitude with the frequency of the external driving force. These steady-state oscillations are described by the *periodic* particular solution of the inhomogeneous differential equation of motion, Eq. (4):

$$\varphi(t) = a \sin(\omega t + \delta). \quad (5)$$

The steady-state oscillations are characterized by definite values of amplitude a and phase lag δ . The phase lag δ is the angular difference between the instants at which the flywheel and the driving rod cross the zero point of the dial (or reach their maximal displacements). Both a and δ depend on ω (the frequency of the external action), ω_0 (the natural frequency), and γ (the damping constant of the oscillator).

The dependencies of a and δ on the external driving frequency, ω , are called the amplitude—frequency and the phase—frequency characteristics of the oscillator. When friction is relatively weak ($\gamma \leq \omega_0$ or $Q \geq 1$), the dependence of the amplitude on the frequency has a *resonance character*: the amplitude increases sharply as ω approaches ω_0 . The graph of the dependence of the steady-state amplitude on the frequency ω is called the *resonance curve*. The greater the quality factor Q , the sharper the peak of the resonance curve, that is, the more pronounced the resonance in the system.

1.6 Forced Oscillations in the Absence Friction

When the driving frequency is sufficiently far from the resonant frequency, we may neglect the influence of friction on the amplitude a and the phase lag δ of steady-state oscillations. That is, we may employ here the idealized frictionless model to describe the behavior of a real system in which there is some friction. (Note that the applicability of a physical model to a real system depends not only on the properties of the system, but also on the problem which we are solving.) Thus, in order to describe steady-state oscillations in the case when $|\omega - \omega_0| \gg \gamma$, we can use Eq. (3), which is valid for a forced linear oscillator in the absence of friction.

As a particular periodic solution describing steady-state oscillations, we try the expression:

$$\varphi(t) = a \sin \omega t. \quad (6)$$

Substituting this expression into Eq. (3), we find that Eq. (6) actually gives a solution of Eq. (3) if the amplitude $a(\omega)$ as a function of frequency ω is:

$$a(\omega) = \frac{\omega_0^2 \phi_0}{\omega_0^2 - \omega^2}. \quad (7)$$

If the driving frequency ω is set equal to zero, Eq. (7) yields $a = \phi_0$: the flywheel is at rest in the displaced equilibrium position. If $\omega \ll \omega_0$, $a \approx \phi_0$: in the case of a very slow motion of the driving rod, the flywheel follows the rod quasistatically. That is, the flywheel remains in the equilibrium position which itself moves slowly alongside the rod. At very low frequencies of the external action, kinematically excited steady-state forced oscillations of the flywheel occur with almost the same amplitude and the same phase as the compelled motion of the driving rod.

Equation (7) shows that as the driving frequency ω is increased, the amplitude of forced oscillations of the flywheel becomes greater. For $\omega \rightarrow \omega_0$ the value a of the amplitude tends to infinity. Consequently, it is inadmissible to ignore friction in the vicinity of resonance (at $\omega \approx \omega_0$). This case is considered below. We note that according to Eq. (7), the value of a becomes negative if $\omega > \omega_0$. The negative sign of a means here that for $\omega > \omega_0$, steady-state oscillations occur with a phase opposite that of the external force: when the rod turns in one direction, the flywheel turns in the other, both reaching their opposite extreme deflections simultaneously.

We can write the solution for $\omega > \omega_0$ in the form of Eq. (5), retaining the positive amplitude a for all frequencies, if we assume a to be equal to the absolute value of the right-hand side of Eq. (7), and the phase shift δ to be equal to $-\pi$. When $\omega < \omega_0$, the phase shift δ in the absence of friction (and also for relatively weak friction, as we shall see later) is zero, so that the flywheel and the rod oscillate in phase. That is, moving in the same direction, they pass the mid-point at the same time, and reach the extremes in their deflections simultaneously. However, as indicated by Eq. (7), the extreme deflection of the flywheel is greater than that of the rod and increases infinitely when the driving frequency approaches the natural frequency (when $\omega \rightarrow \omega_0$).

1.7 The Resonance Curve

In the vicinity of resonance (at driving frequencies ω that satisfy the condition $|\omega - \omega_0| \leq \gamma$), it is necessary to take friction into account in the differential equation of forced oscillations. That is, we need to solve Eq. (4). Steady-state forced oscillations are described by its particular periodic solution. We can write this solution in the form $\varphi(t) = a \sin(\omega t + \delta)$ (see Eq. (5)). Substituting that solution into Eq. (4), we can search for the values of a and δ for which the function $a \sin(\omega t + \delta)$ satisfies Eq. (4). Leaving the derivations as an exercise, we give here the final expressions for the amplitude a and the phase shift δ :

$$a(\omega) = \frac{\omega_0^2 \phi_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2 \omega^2}}, \quad \tan \delta = -\frac{2\gamma\omega}{\omega_0^2 - \omega^2}. \quad (8)$$

The graphs of the dependence of the amplitude on the frequency $a(\omega)$ (the resonance curves) and the phase lag on the frequency $\delta(\omega)$ for different values of the quality factor Q are shown in upper and lower parts of figure 2, respectively. The maximal value of the amplitude of steady oscillations occurs at the *resonant frequency* ω_{res} :

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\gamma^2}. \quad (9)$$

This expression for ω_{res} is valid if friction is not too large, that is, if $\omega_0 > \sqrt{2}\gamma$. When friction is sufficiently small, that is, if $\gamma \ll \omega_0$ or $Q \gg 1$, we see from Eq. (9) that

$$\omega_{\text{res}} = \omega_0 \sqrt{1 - \frac{2\gamma^2}{\omega_0^2}} \approx \omega_0 \left(1 - \frac{\gamma^2}{\omega_0^2}\right) = \omega_0 \left(1 - \frac{1}{4Q^2}\right).$$

That is, the resonant frequency nearly coincides with the natural frequency ω_0 : the value ω_{res} differs from ω_0 only by a term of the second order in the small parameter γ/ω_0 . For example, if $Q = 10$ (moderate friction), the resonant frequency differs from the natural frequency only by 0.25%.

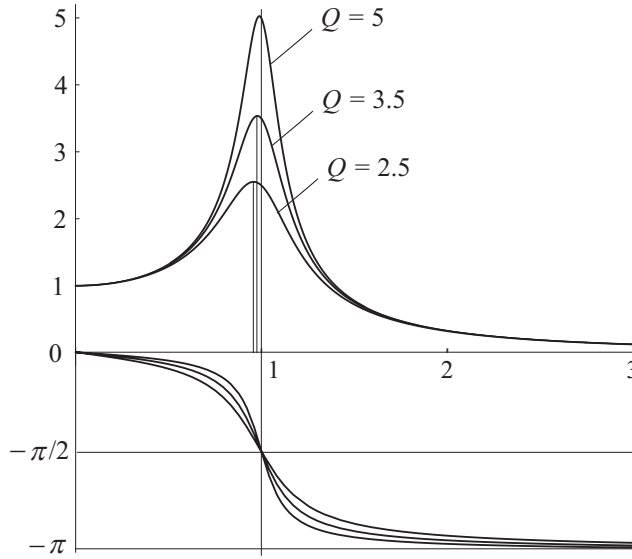


Figure 2: Resonance curves of a linear oscillator

The amplitude of steady oscillations at resonance is determined by the expression:

$$a_{\text{max}} = \frac{\omega_0^2 \phi_0}{2\gamma \sqrt{\omega_0^2 - \gamma^2}} \approx \frac{\omega_0 \phi_0}{2\gamma} = Q \phi_0. \quad (10)$$

We see from Eq. (10) that the amplitude a_{max} of steady-state oscillations at resonance is approximately Q times greater than the amplitude ϕ_0 of the driving rod (provided that the quality factor Q is not too low). In other words, the amplitude a_{max} of steady-state oscillations at resonance is Q times greater than the amplitude $a(0)$ of steady-state oscillations at a very low driving frequency ω (at slow oscillations of the rod).

We note that the resonant properties of a linear oscillator under forced oscillations and the damping of its natural free oscillations are characterized by the same quantity, the quality factor Q .

When there is no friction, Eq. (8) shows that the amplitude of the flywheel during steady-state forced oscillations is greater than the amplitude ϕ_0 of the rod at all frequencies ω between zero and the boundary value $\sqrt{2}\omega_0$. When the frequency ω of the driving force exceeds the natural frequency ω_0 by more than $\sqrt{2}$ times, the amplitude of steady-state forced oscillations is smaller than ϕ_0 and approaches zero as the frequency ω increases further. In this range of frequencies the dynamic effect of an external driving force is less than the static effect of a constant force of the same magnitude. The physical cause of such behavior is the inertia of the flywheel: when the driving frequency of the rod is considerably greater than the natural frequency of the flywheel, the massive flywheel cannot follow the rapid motion of the rod. The same is true also in the presence of moderate friction, except that the bounding frequency is slightly smaller than $\sqrt{2}\omega_0$, as can be seen from the graphs of $a(\omega)$ plotted in figure 2 using Eq. (8).

Equation (8) for the phase shift δ and the corresponding graphs in the lower part of figure 2 show that steady-state forced motion always lags behind the driving force since δ is always negative. Far from resonance at $\omega < \omega_0$ this lag is nearly zero, and the flywheel oscillates nearly in phase with the exciting rod. When $\omega = \omega_0$, steady-state oscillations of the flywheel lag in phase behind oscillations

of the exciting rod by a quarter of the period ($\delta = -\pi/2$) for all values of friction. In this case the displacement of the flywheel is greatest when the displacement of the rod is zero, and vice versa. When ω is much greater than ω_0 , the phase shift δ approaches $-\pi$. That is, the lag is nearly 180° . In this case the flywheel and the exciting rod always rotate in opposite directions.

If there is no friction, Eq. (8) indicates that the phase lag is either 0 (for $\omega < \omega_0$) or 180° (for $\omega > \omega_0$). That is, when $\omega = \omega_0$, there is an abrupt change from motion of the flywheel exactly in phase with that of the rod to motion in which they oscillate exactly in opposite phase. (In the absence of friction, the amplitude of the flywheel at the transition is infinite.) In the presence of friction, the transition from in-phase steady-state oscillations of the flywheel and the rod to opposite-phase steady-state oscillations takes place gradually over a range of frequencies centered about ω_0 . The width of this range, as can be seen from figure 2, is proportional to the damping constant γ .

1.8 Resonance of the Angular Velocity

In steady-state oscillations under a sinusoidal force, the angular velocity of the flywheel $\dot{\varphi} = a\omega \cos(\omega t + \delta)$ changes in time harmonically with the frequency ω of the external driving force. The expression for the amplitude of the angular velocity $\Omega = \dot{\varphi}$ differs from Eq. (8) for the amplitude $a(\omega)$ of these oscillations by an additional factor ω :

$$\Omega(\omega) = \omega a(\omega) = \frac{\omega_0^2 \phi_0}{\sqrt{(\omega_0^2/\omega - \omega)^2 + 4\gamma^2}}. \quad (11)$$

Dependence of the velocity amplitude on the frequency is shown in the upper part of figure 3.

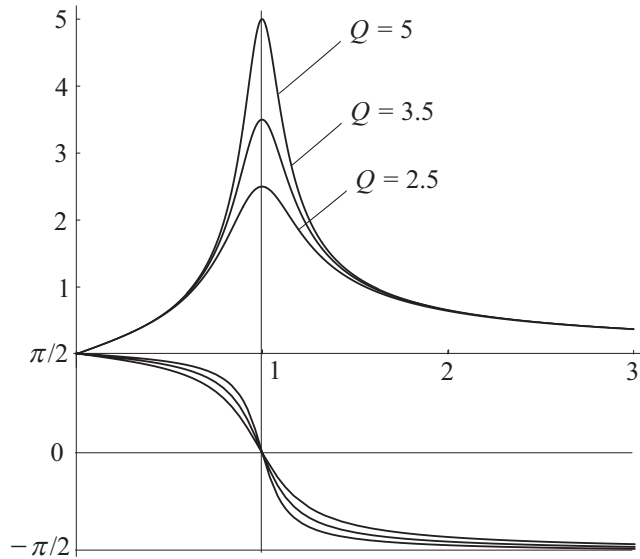


Figure 3: Resonance curves for the velocity of a linear oscillator

As we can see from Eq. (11), the maximum of the resonance curve for the angular velocity is located at $\omega = \omega_0$ independently of the damping factor γ . Therefore resonance of the angular velocity occurs at the value of the driving frequency ω which exactly equals the natural frequency ω_0 of the oscillator for both weak and strong friction. On the other hand, resonance of the angular displacement occurs at $\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\gamma^2}$.

The lower part of figure 3 shows the dependence of the phase shift between the driving rod and the angular velocity. At resonance (at $\omega = \omega_0$) this phase shift equals zero: the driving rod oscillates in

phase with the velocity. This means that at resonance the energy is transmitted from the exciter to the oscillator during the whole period.

1.9 Energy Transformations

Though the amplitude is constant in steady-state forced oscillations, the total energy of the oscillator is constant only on the average. During one quarter of a cycle, energy is transmitted from the driving rod to the oscillator, and during the next quarter cycle, energy is transferred back from the oscillator to the external source driving the rod. In contrast to free oscillations, not only do the kinetic and potential energies oscillate, but so also does their sum, the total mechanical energy. The total energy oscillates with double the frequency of the external force.

In our discussion of steady-state energy transformations, we pay special attention to the distinguishing characteristics of the energy transfer for kinematic excitation of forced oscillations modeled in these computer simulations, and for dynamic excitation, for which the external torque is a given time-dependent quantity. We have already mentioned these distinctions above. Here we discuss them in detail.

When the dynamic mode of excitation is used, an external torque is applied directly to the flywheel. With one end of the spiral spring fixed, the deformation of the spring (the amount of twisting from its unstrained state) is determined by the angular displacement φ of the flywheel from the midpoint (from the equilibrium position). Thus the potential energy of the spring is given by

$$E_{\text{pot}} = \frac{1}{2}D\varphi^2(t) = \frac{1}{2}Da^2 \sin^2(\omega t + \delta) = \frac{1}{4}J\omega_0^2 a^2 [1 - \cos 2(\omega t + \delta)]. \quad (12)$$

The kinetic energy of the oscillating flywheel is independent of the mode of excitation and is given by the following expression

$$E_{\text{kin}} = \frac{1}{2}J\dot{\varphi}^2(t) = \frac{1}{2}J\omega^2 a^2 \cos^2(\omega t + \delta) = \frac{1}{4}J\omega^2 a^2 [1 + \cos 2(\omega t + \delta)]. \quad (13)$$

It is seen that, while the oscillator executes steady-state forced oscillations with the frequency ω , the values of its potential and kinetic energies are oscillating harmonically with the frequency 2ω in opposite phases with respect to one another. The ratio of their maximal (and average) values is equal to the squared ratio of the natural frequency to the driving frequency:

$$\frac{\langle E_{\text{pot}} \rangle}{\langle E_{\text{kin}} \rangle} = \frac{\omega_0^2}{\omega^2}. \quad (14)$$

Hence, when $\omega < \omega_0$, the potential energy predominates on the average over the kinetic energy. In particular, when $\omega \ll \omega_0$, the spring is twisted quasistatically, and nearly all the energy of the oscillator is the elastic potential energy of the strained spring. On the other hand, for external frequencies which exceed the natural one ($\omega > \omega_0$), the kinetic energy predominates over the potential energy.

The peculiarities of energy transformations for the kinematic excitation of oscillations are related to the fact that the equilibrium position of the flywheel (and its potential well as a whole) is displaced when the driving rod is turned. The deformation of the spring in this case is determined by the difference in the angles $\varphi(t)$ and $\phi(t)$. The expression for its potential energy then takes the form:

$$E_{\text{pot}} = \frac{1}{2}D(\varphi - \phi)^2. \quad (15)$$

When the frequency of the rod is much less than the resonant frequency, the flywheel moves as though it were attached to the slowly moving rod. The flywheel remains close to its equilibrium position, which

is displaced by the driving rod. The spring remains nearly unstrained and its potential energy is nearly zero. In other words, the oscillator is always located near the bottom of its potential well, which slowly oscillates alongside the rod. Therefore, at low frequencies of the driving mechanism, the kinetic energy predominates over the potential energy, in direct contrast to the case of dynamic excitation.

When the driving frequency is large compared to the natural frequency, the inertia of the flywheel diminishes the response the flywheel is able to make to the displacement of the equilibrium position by the external source, and so the flywheel oscillates with a relatively small amplitude. At high driving frequencies, the amplitude of the steady-state deflections of the flywheel from its central position is much smaller than the amplitude ϕ_0 of the angular displacement of the driving rod. The spring is twisted back and forth through approximately the angle ϕ_0 , while the angular velocity of the flywheel remains relatively small. Hence the elastic potential energy arising from the deformation of the spring predominates over the kinetic energy of the flywheel, again in direct contrast to the case of dynamic excitation.

When friction is small ($\gamma \ll \omega_0$), the ratio of the average value of the potential energy to the average value of the kinetic energy for kinematic excitation can be found from Eq. (15) and Eq. (5) for $\varphi(t)$ and Eq. (1) for $\phi(t)$:

$$\frac{\langle E_{\text{pot}} \rangle}{\langle E_{\text{kin}} \rangle} = \frac{\omega^2}{\omega_0^2}. \quad (16)$$

Comparing Eq. (16) with Eq. (14), we see that the ratios of the average value of the potential energy to the average value of the kinetic energy for dynamic and kinematic modes of excitation of forced oscillations are inverses of one another, in agreement with the general discussion above.

1.10 Transient Processes

The amplitude and the phase of steady-state oscillations do not depend on the initial conditions. Loosely speaking, during the transient process the oscillator eventually “forgets” them. We should keep in mind that steady-state oscillations are described by Eq. (5), which is the *periodic particular solution* to the inhomogeneous differential equation, Eq. (4). The graphs of the amplitude and phase versus the driving frequency (the amplitude—frequency and the phase—frequency characteristics of a linear oscillator), displayed in figure 2, refer to this particular solution and are valid only for steady-state oscillations.

The initial conditions, namely the initial angle of deflection $\varphi(0)$ and the initial angular velocity $\dot{\varphi}(0)$, are influential only during the *transient process*. During the transition, natural damped oscillations are superimposed on the steady-state forced oscillations. The effects of the natural oscillations disappear once the steady-state oscillations have been established.

Mathematically a transient process is represented by the *general solution* to the inhomogeneous equation, Eq. (4). This complete solution is given by:

$$\varphi(t) = a \sin(\omega t + \delta) + C e^{-\gamma t} \cos(\omega_1 t + \theta). \quad (17)$$

The first term on the right is the periodic particular solution, Eq. (5), to the inhomogeneous equation, Eq. (4). The second term on the right, called the *transient term*, is the general solution to the corresponding homogeneous equation, namely Eq. (4) in which the right-hand side is zero. This solution of the homogeneous equation is the contribution of damped natural oscillations to the transient process. The frequency ω_1 of this term nearly equals the natural frequency ω_0 provided the friction is not too large:

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2} = \omega_0 \sqrt{1 - \frac{\gamma^2}{\omega_0^2}} \approx \omega_0 \left(1 - \frac{\gamma^2}{2\omega_0^2} \right) = \omega_0 \left(1 - \frac{1}{8Q^2} \right). \quad (18)$$

The fractional difference $(\omega_0 - \omega_1)/\omega_0$ in most cases of practical importance is so small that we can neglect it and assume that $\omega_1 = \omega_0$. Indeed, if $Q = 5$, the fractional difference is only 0.5%: $(\omega_0 - \omega_1)/\omega_0 = 0.005$.

The transient term of the general solution contains two arbitrary constants, C and θ . Their values depend on the initial conditions, namely on the angular displacement and the angular velocity of the flywheel at the instant the external force begins to act.

Thus the transient process is described by a superposition of two oscillations: a sinusoidal steady oscillation with a constant amplitude a and a frequency ω of the driving force, and a damped natural oscillation with the frequency $\omega_1 \approx \omega_0$ and a decaying amplitude. In principle, the transient process continues indefinitely, but in practice it is considered completed after the transient term essentially dies out, that is, after about Q cycles of natural oscillations. (We recall that Q is the dimensionless quality factor, given by $Q = \omega_0/2\gamma$.) Generally, the smaller the friction, the longer the transient process lasts. However, as we shall see in the next section, it is possible to choose initial conditions such that there is no transient term and no transient process.

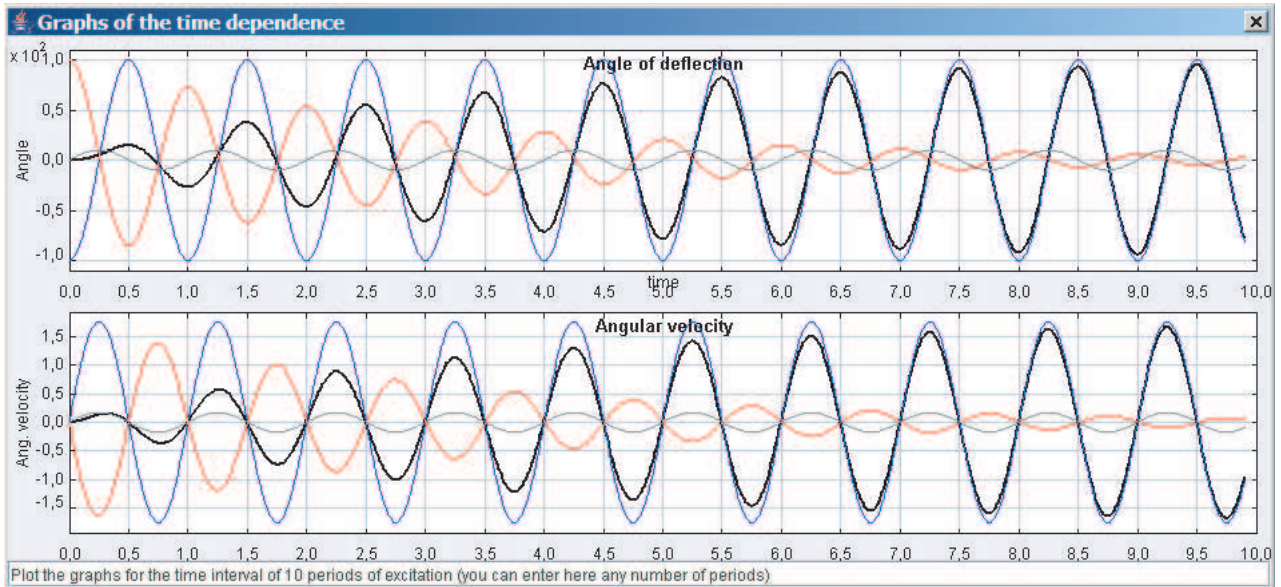


Figure 4: Decomposition of a transient process into the sum of steady-state forced oscillations and damped natural oscillations (time-dependent graphs of the angular displacement and angular velocity).

There is an option in the computer simulation program which allows to display the plots of the two simple oscillations which constitute the transient process while simultaneously plotting their superposition. An example of the decomposition of a transient process into the two separate simple components (the steady-state sinusoidal oscillation with a constant amplitude and the frequency of the external driving action and the decaying natural transient oscillation) is shown in figure 4. The graphs correspond to resonance (the driving frequency equals the natural one).

1.11 Initial Conditions which Eliminate a Transient

As noted above, it is possible to choose the initial conditions in such a way that there is no transient process, that is, so that steady-state oscillations appear immediately upon activation of the external force. To find these conditions, we observe that if we take as the angular displacement and the angular velocity

at $t = 0$ the corresponding values which characterize the steady-state oscillations, the initial conditions are satisfied by the steady-state itself, without the addition of natural oscillations.

It follows from Eq. (5) that the required initial angular deflection $\varphi(0)$ is $a \sin \delta$, and the required initial angular velocity $\dot{\varphi}(0)$ is $a\omega \cos \delta$, where a and δ are the amplitude and the phase of steady-state oscillations given by Eq. (8). Since the steady-state term in Eq. (17) satisfies the initial conditions by itself, the transient term vanishes, leaving only the steady-state, so that C in Eq. (17) must be zero. In other words, if $\varphi(0) = a \sin \delta$ and $\dot{\varphi}(0) = a\omega \cos \delta$, no transient natural oscillation arises when the external force begins to act, and there is no contribution to the motion from the homogeneous equation.

Transient processes make the phenomenon of forced oscillations much more complicated than are simple harmonic steady-state oscillations. In many cases these transient processes are important and interesting in themselves. They are worth considerable attention in your work with the simulation computer program.

1.12 Forced Oscillations from Rest at Resonance

We next restrict our study of transient processes to the case in which the initial conditions are zero. That is, we consider forced oscillations of a flywheel at rest in the equilibrium position at the instant the driving force begins to operate:

$$\varphi(0) = 0, \quad \dot{\varphi}(0) = 0. \quad (19)$$

At $t = 0$, the driving rod starts moving according to:

$$\phi(t) = \phi_0 \sin \omega t. \quad (20)$$

Let us first consider the case of an oscillator damped by relatively weak friction ($\gamma \ll \omega_0$). Because there is little friction, the resonant frequency is very nearly the natural frequency ω_0 . For the case in which the driving frequency ω is set equal to ω_0 , it follows from Eqs. (8) that the periodic particular solution which describes the steady-state oscillations is given by:

$$\varphi(t) \approx \frac{\omega_0}{2\gamma} \phi_0 \sin \left(\omega_0 t - \frac{\pi}{2} \right) = -Q \phi_0 \cos \omega_0 t. \quad (21)$$

In this case the amplitude of oscillation of the flywheel is greater than the amplitude of oscillation of the rod by the factor Q , and the phase lag is $-\pi/2$. That is, the oscillations of the flywheel are one quarter of a cycle behind the oscillations of the driving rod.

The transient term in Eq. (17) with $\omega_1 = \omega_0$ is next added to Eq. (21) to obtain a complete solution:

$$\varphi(t) = -Q \phi_0 \cos \omega_0 t + C e^{-\gamma t} \cos(\omega_0 t + \theta). \quad (22)$$

The arbitrary constants C and θ are determined by satisfying the initial conditions, Eqs. (19). In the case of weak friction, when $\gamma \ll \omega_0$, the exponential factor $e^{-\gamma t}$ in Eq. (22) changes little over many oscillations and when taking the time derivative of Eq. (22), can be considered approximately constant for long durations:

$$\dot{\varphi}(t) \approx Q \phi_0 \omega_0 \sin \omega_0 t - C e^{-\gamma t} \omega_0 \sin(\omega_0 t + \theta). \quad (23)$$

Requiring that $\dot{\varphi}(0)$ be zero implies that $\theta = 0$, and requiring that $\varphi(0)$ be zero implies that $C = Q \phi_0$. Hence, if $Q \gg 1$ and $\omega = \omega_0$, the solution of differential equation of motion, Eq. (5), satisfying the zero initial conditions of Eq. (19) is:

$$\varphi(t) = -Q \phi_0 (1 - e^{-\gamma t}) \cos \omega_0 t = -b(t) \cos \omega_0 t, \quad (24)$$

where

$$b(t) = Q\phi_0(1 - e^{-\gamma t}). \quad (25)$$

This superposition of forced and slowly decaying natural oscillations, each with the frequency ω_0 , can be considered as a single nearly harmonic oscillation with the frequency ω_0 and an amplitude $b(t)$ which slowly increases with time, asymptotically approaching the steady-state value, $Q\phi_0$. The graph of such transient process at resonance (together with graphs of the angular displacement and the angular velocity) for the zero initial conditions) is shown by a thick line in figure 4.

By convention, the duration of this transient process is assumed to be equal to the time of damping τ ($\tau = 1/\gamma$), which is the time during which natural free oscillations of the transient term fade away.

This gradual growth of the amplitude $b(t)$ and its gradual asymptotic approach to a maximum value during the transient process at the resonant frequency can be easily explained on the basis of energy transformations. At resonance a definite phase relation establishes between the oscillation of the flywheel and that of the rod. Namely, the rod oscillates in phase with the angular velocity of the flywheel. This phase relation provides the conditions that are favorable for the transfer of energy from the rod to the oscillator. For large values of the quality factor Q , the amplitude of the flywheel eventually increases to the value $Q\phi_0$, which considerably exceeds (by a factor of Q) the amplitude ϕ_0 of the rod. The greater the quality factor Q , the greater the energy eventually stored by the oscillator and the greater the number of cycles required to transfer this energy to the oscillator by a weak driving force. The growth of the amplitude decreases to zero when the velocity-dependent friction dissipates any further addition of energy from the driving rod. During steady-state oscillations, the energy dissipated over a cycle equals the energy transmitted to the oscillator from the external source driving the rod.

In the case of weak friction the duration of the transient process is large compared with the period of oscillations: $\tau \gg T_0$. The amplitude of oscillation of the flywheel, initially at rest, increases over many cycles. The early growth of the amplitude occurs almost linearly with time. This behavior is evident from Eq. (25), in which we let $\gamma t \ll 1$. Expanding the exponential in a power series and keeping only the linear term, we have:

$$b(t) = Q\phi_0(1 - e^{-\gamma t}) = \frac{\omega_0}{2\gamma}\phi_0(1 - e^{-\gamma t}) \approx \frac{1}{2}\phi_0\omega_0 t. \quad (26)$$

Cancellation of the damping constant γ in the latter expression means that the linear growth of the amplitude during the early stage of the transient process (while $\gamma t \ll 1$, or $t \ll QT_0$) occurs just as though friction were absent. In the idealized case of the complete absence of friction, such linear growth of the amplitude would continue indefinitely.

Thus, steady-state oscillations are impossible for a frictionless oscillator at the resonant frequency. In this case, the particular solution of the inhomogeneous differential equation of motion (Eq. (4), with $\gamma = 0$ and $\omega = \omega_0$), rather than being periodic, increases without limit: $\varphi(t) = \frac{1}{2}\phi_0\omega_0 t \cos \omega_0 t$. To obtain the general solution containing two arbitrary constants, we must also include the general solution of the corresponding homogeneous equation, which in this case describes natural oscillations of constant amplitude. Determining the constants from the zero initial conditions, we find:

$$\varphi(t) = \frac{1}{2}\phi_0(\omega_0 t \cos \omega_0 t - \sin \omega_0 t). \quad (27)$$

The amplitude of this oscillation with the frequency ω_0 is $\frac{1}{2}\phi_0\sqrt{(\omega_0 t)^2 + 1} \approx \frac{1}{2}\phi_0\omega_0 t$. Thus, during the resonant transient process in the absence of friction the amplitude of the flywheel, initially at rest, increases steadily at almost a constant rate. This rate is the same as the early rate in the presence of weak friction, because during the early stage of the transient process the growth of mechanical energy is almost unimpeded by the velocity-dependant friction.

The unlimited growth of the amplitude at resonance predicted by Eq. (27) means, as noted above, that in this frictionless system steady-state oscillations are not possible when $\omega = \omega_0$. In fact this growth means that in conditions of resonance the idealized frictionless model does not work; that is, we can not use the model for description of resonance in a real system no matter how small the friction.

In a real system, at sufficiently large amplitudes, either friction dissipates the added energy (so that in the model of the system it is necessary to take friction into account) or the amplitude of the oscillation increases beyond the linear limit of the restoring force of the spring so that Hooke's law fails. In the latter case, the nonlinear dependence of the restoring force on the angle of deflection changes the period of natural oscillations as the amplitude grows, and resonance is destroyed. For a nonlinear system, the growth of the amplitude is restricted even in the absence of friction: the resonant conditions become violated at large amplitudes. To say which of these reasons (friction or nonlinearity) restricts the growth of the amplitude in a real physical system, we must know more about the properties of the system.

1.13 Transient Processes Near Resonance

If the driving frequency ω is near the frequency ω_1 of damped natural oscillations, then during the transient process, while the natural oscillations have not yet damped away, we observe the addition of two oscillations with slightly different frequencies, ω and ω_1 . As already mentioned, the frequency ω_1 nearly equals the natural frequency ω_0 provided the friction is not too large [see Eq. (18)], so that we need not distinguish between them here.

The oscillation resulting from this addition is *modulated*: its amplitude slowly alternately increases and decreases with a *beat frequency* equal to the difference $|\omega - \omega_0|$ between the driving and natural frequencies. The external force first drives the oscillator to amplitudes which exceed the steady-state value; then the accumulated phase shift between the oscillations of the flywheel and those of the driving rod causes energy to flow back from the oscillator to the source of the external action, and the amplitude of the flywheel decreases. These cycles of *transient beats* (of slow variations of the amplitude) are repeated over and over until the damped natural oscillations die out.

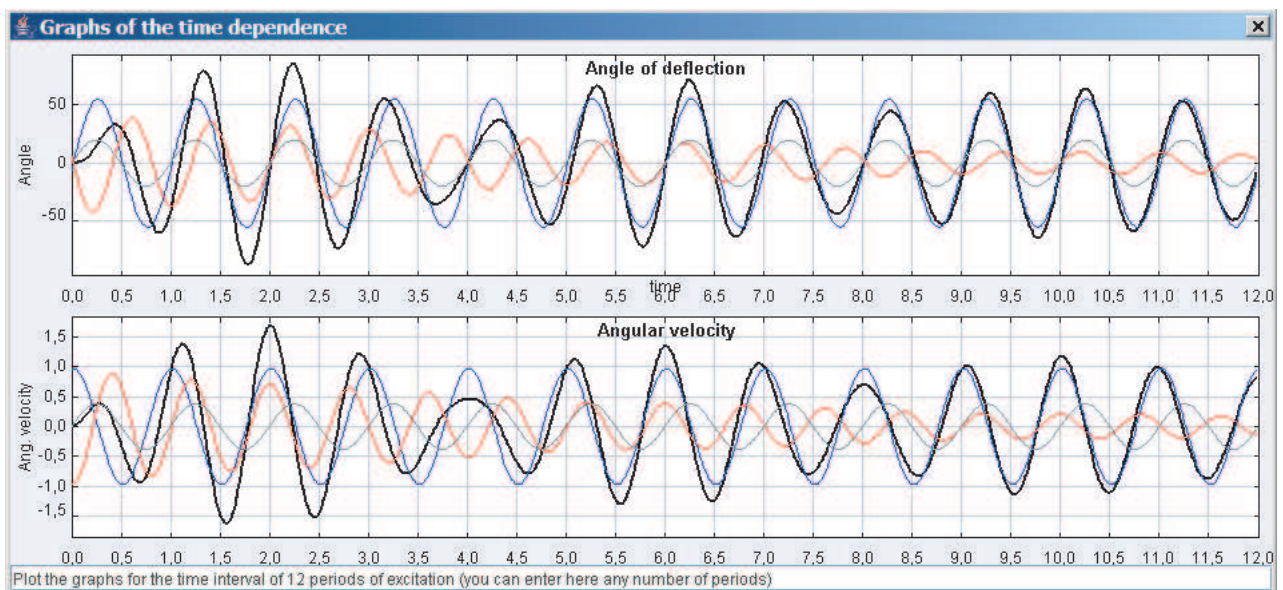


Figure 5: Gradually fading beats during a transient process near the resonant frequency ($\omega = 0.8\omega_0$). The graphs show the angular displacement and velocity for the zero initial conditions.

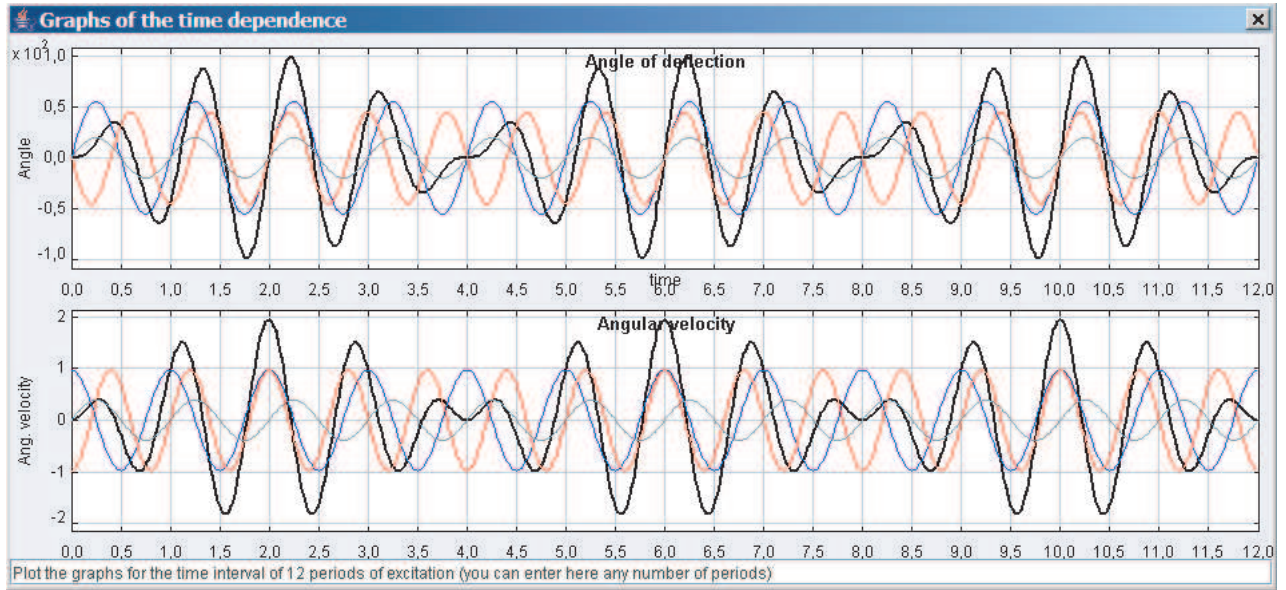


Figure 6: Non-fading beats in the absence of friction during a transient process near the resonant frequency ($\omega = 0.8\omega_0$) for the zero initial conditions.

In the presence of viscous friction, the modulation of the amplitude is gradually diminished as damping decreases the contribution of the transient oscillations with the natural frequency. Figure 5 displays graphs of such fading transient beats for the case in which the flywheel is initially at rest. In this example, four cycles of the driving force at the frequency $\omega = 0.8\omega_0$ occur during five cycles of natural oscillations at the frequency ω_0 . (On the graph, the vertical hatch marks correspond to the external period T .) Hence one period, $T_b = 2\pi/|\omega - \omega_0|$, of the beat cycle occurs during four periods, $T = 2\pi/\omega$, of the driven (steady-state) cycle and during five periods, $T_0 = 2\pi/\omega_0$, of natural oscillations.

In the absence of friction, steady-state oscillations with the frequency ω occur in phase with the external force at $\omega < \omega_0$, or 180° out of phase at $\omega > \omega_0$. Their amplitude a is determined by Eq. (7). The other contribution in the transient process is given by oscillations with the natural frequency ω_0 . At $\gamma = 0$ these natural oscillations are not damped. Thus, in this case an addition of two harmonic oscillations with slightly different frequencies ω and ω_0 and *constant* amplitudes occurs after the external force is activated. For the zero initial conditions the amplitude of the natural oscillation equals $-a(\omega/\omega_0)$. The resulting beat oscillation is *pure* in the sense that the beats do not decay. We can consider the beat oscillation as an oscillation with a mean frequency $(\omega + \omega_0)/2$ and a slowly periodically varying amplitude. The graphs of such pure non-fading beats are shown in figure 6. The envelope of the oscillations changes sinusoidally, periodically equaling zero. The transient process lasts indefinitely, and so there is no steady-state oscillation in the absence of friction (at $\gamma = 0$).

1.14 Transient Processes Far from Resonance

Here we consider nonresonant cases in which the external frequency ω is much different from the natural frequency ω_0 .

If the external driving frequency is much less than the natural frequency ($\omega \ll \omega_0$), the equilibrium position of the flywheel (in which the spring is unstrained) slowly moves back and forth alongside the rod. Simultaneously, the flywheel executes relatively rapid damped oscillations at its natural frequency about this slowly moving equilibrium position. As a result, these gradually fading rapid natural os-

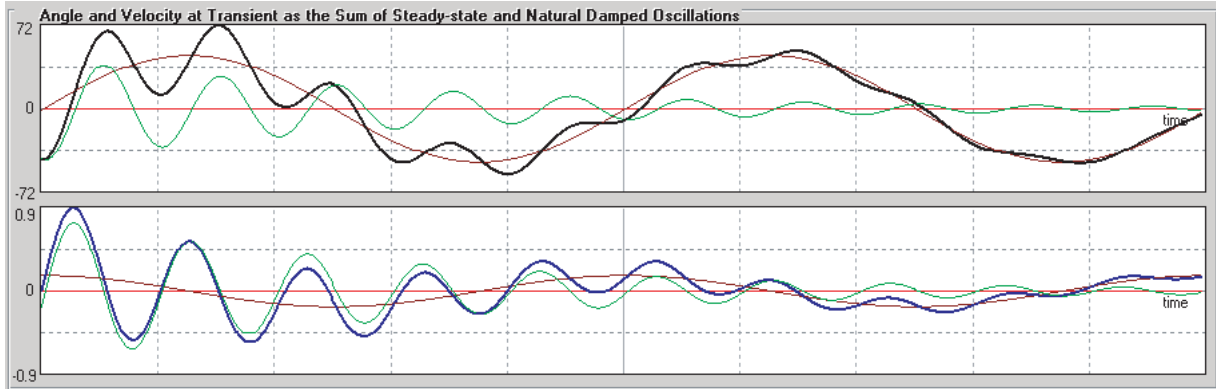


Figure 7: The angular displacement and angular velocity during a transient process at a low driving frequency ($\omega \ll \omega_0$)

cillations, superimposed on the slow steady-state forced oscillations of a constant amplitude, produce a pattern of motion like that shown in the graph in figure 7. When these natural oscillations die out, the plot evolves into a pure, undistorted sine wave corresponding to steady-state oscillations whose frequency is the slow driving frequency ω .

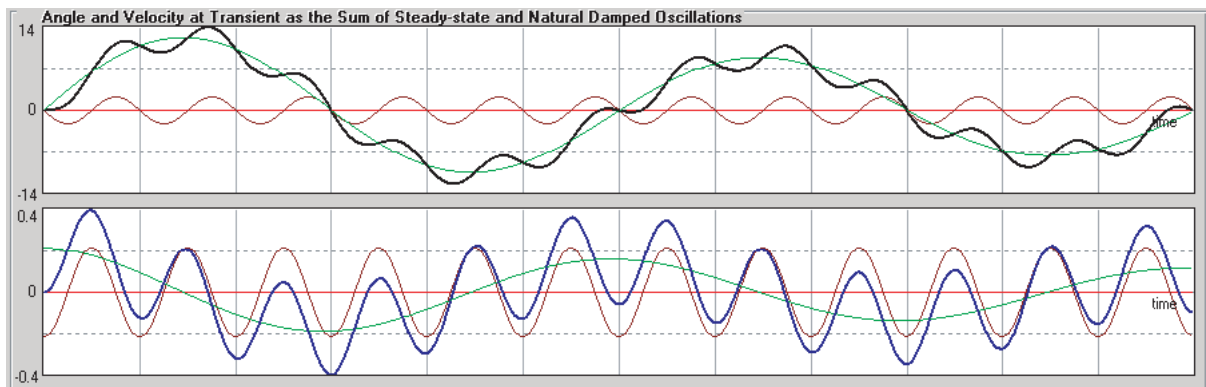


Figure 8: The angular displacement and angular velocity during a transient process at a high driving frequency ($\omega \gg \omega_0$)

In the opposite case in which the exciting rod oscillates with a high frequency ($\omega \gg \omega_0$), relatively rapid forced oscillations at the frequency ω and a constant amplitude occur about a middle position, which, during the transient process, slowly executes damped oscillations at the natural frequency ω_0 . After these oscillations have died out, only the rapid forced oscillations of a constant amplitude remain. These rapid steady-state oscillations occur symmetrically about the value $\phi = 0$, i.e., about the central position of the rod. This case is illustrated in figure 8.

1.15 Transient Processes and the Phase Trajectory

The equation of motion describing forced oscillations, Eq. (4), is explicitly time-dependent. In this equation a given function of time, $\phi(t) = \phi_0 \sin \omega t$, describes the forcing periodic motion of the driving rod. Thus the mechanical state of the system under consideration is determined by the three quantities: φ , $\dot{\varphi}$, and t .

In order to display all of the characteristics of the mechanical state of an oscillator acted upon by a given time-dependent external force, we add a temporal dimension to the phase plane $(\varphi, \dot{\varphi})$. This temporal third dimension is introduced by erecting a time axis perpendicular to the phase plane. In plotting the solution to Eq. (4), the computer simulation program traces the projection onto the plane $(\varphi, \dot{\varphi})$ of a twisted three-dimensional phase trajectory arising from forced oscillations.

To obtain a clear graphic representation of the entire transient process, we mark positions of the representative point on this phase trajectory at equal time intervals, at the moments when the rod, moving from left to right, crosses the zero point of the dial.

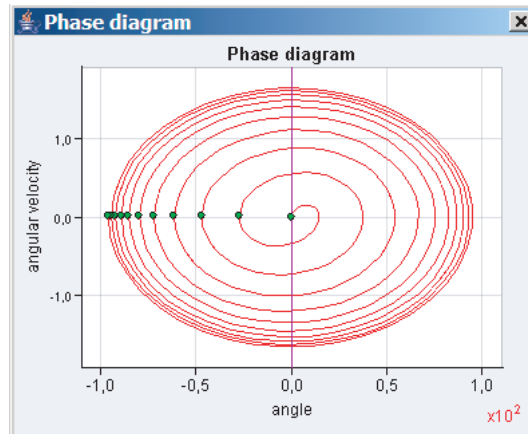


Figure 9: Phase trajectory with Poincaré sections for the transient at resonance swinging of the oscillator.

These points on the phase trajectory show the mechanical state of the system at times equal to integral multiples of the period of oscillation, $T = 2\pi/\omega$, of the external force. Such points are called *Poincaré sections*. The simulation program displays the phase trajectory with Poincaré sections (figure 9) when you open the window “Phase diagram” by using a relevant button on the control panel of the program.

Since steady-state oscillations have the period of the external force and a constant amplitude, the corresponding three-dimensional phase trajectory intersects all the planes $t = T, 2T, \dots, nT$ at the same values of φ and $\dot{\varphi}$. Thus the projections of Poincaré sections onto the plane $(\varphi, \dot{\varphi})$ at the times $t_n = nT$ coincide.

However, for a transient process, projected Poincaré sections form a set of points in the plane $(\varphi, \dot{\varphi})$ which condense gradually to the point $\varphi = a \sin \delta, \dot{\varphi} = a\omega \cos \delta$, the projected Poincaré section for the steady-state oscillations.

At the resonant frequency ($\omega = \omega_0$), the oscillation of the flywheel lags behind that of the rod by a quarter cycle ($\delta = \pi/2$), and the coordinates of the limiting condensing point of the Poincaré sections for the transient process are $\varphi = -a, \dot{\varphi} = 0$. If the initial angular deflection and angular velocity of the flywheel are zero, the rod starts to move when the flywheel is at rest in the equilibrium position. Then all the projections of the Poincaré sections lie on the abscissa axis of the phase plane, starting at the origin and gradually approaching the above-mentioned condensing point, $\varphi = -a, \dot{\varphi} = 0$ (see figure 9).

2 Questions, Problems, Suggestions

The simulation of forced oscillations of a torsion spring pendulum in the computer program is based on the numerical integration of differential equation, Eq. (4), by means of the Runge—Kutta method of the fourth order. Although this linear differential equation can be solved analytically, the analytic solution

is not used in this simulation program. The agreement between the analytic theoretical predictions and the observed behavior of the linear oscillator in these simulations is an indicator of the reliability of the numerical method. Thus we get confidence in the reliability of the simulations of nonlinear systems in other programs in the package “Physics of Oscillations,” because these simulations are based on the same numerical method. This confidence is not without value since analytic solutions are unavailable for nonlinear systems whose behavior in simulations is often hard to reconcile with common sense.

2.1 Steady-state Forced Oscillations

In order to display steady-state oscillations without a preliminary transient process (when the option “Show steady-state” is chosen), the simulation program automatically sets the initial conditions to be $\varphi(0) = a \sin \delta$ and $\dot{\varphi}(0) = a\omega \cos \delta$ (overriding any initial conditions you may have entered). These initial conditions provide steady-state oscillations immediately after the external force is activated. The values of the amplitude a and the phase δ are calculated in the program from Eq. (8) using the values of the external frequency ω and the quality factor Q you have entered.

1.1 Steady-state Forced Oscillations without Friction.

(a) Setting properties of the system, choose full absence of friction. In this case, the transient process lasts indefinitely, so that steady-state oscillations do not establish. How can you explain the physical sense of the analytical solution that describes steady-state forced oscillations in the absence of friction? Is this solution applicable to a real system? If so, what conditions must be satisfied in order that it be possible in a real system to observe the motion described by this analytical solution?

(b) Convince yourself that for driving frequencies less than the natural frequency ($\omega < \omega_0$), the steady-state sinusoidal oscillations of the flywheel occur exactly in phase with the oscillations of the driving rod. At what frequency is the amplitude of the flywheel twice that of the rod? Calculate this frequency and verify your answer with a simulation experiment.

(c) Convince yourself that for driving frequencies greater than the natural frequency ($\omega > \omega_0$), the phase of the steady-state oscillations of the flywheel is exactly opposite that of the rod. At what value of the driving frequency ($\omega > \omega_0$) is the amplitude of the flywheel again twice that of the rod? At what driving frequency are these amplitudes equal? At what frequency is the amplitude of the flywheel one half that of the rod?

1.2* Transformations of Energy for Steady-state Oscillations.

(a) Using the plots of potential, kinetic, and total mechanical energy, find out during which parts of the driving cycle is energy transmitted from the rod to the oscillator. Give a physical explanation of this direction of energy transfer. During which parts of the cycle is energy transferred back from the oscillator to the external source?

(b) For the kinematic excitation of forced oscillations, what is the ratio of the average value of the potential energy to the average value of the kinetic energy for each of those values of frequency for which the amplitude of the flywheel is twice that of the rod? For the case when the amplitudes are equal? For the case when the amplitude of the flywheel equals one half that of the rod? Compare the values observed on the experimental plots with your calculated values.

1.3 The Amplitude and Phase of Steady-state Oscillations.

Examine steady-state oscillations in the presence of friction. Input some moderate value of the quality factor, say, $Q = 5$.

(a) Evaluate the percentage shift of the resonant frequency from the natural frequency.

(b) What is the ratio of the amplitude of steady-state oscillation of the flywheel to the amplitude of the rod at resonance?

(c) What is the phase lag of the oscillation of the flywheel relative to the phase of the rod at the resonant frequency and at a driving frequency equal to 0.8 of the resonant value?

Answer the same questions for $Q = 20$.

1.4 Peculiarities of the Kinematic Excitation.** In the case of the dynamic excitation of oscillations by a given force whose value is independent of the position of the flywheel, the ratio of the average potential energy to the average kinetic energy equals $(\omega_0/\omega)^2$, so that for low frequencies the potential energy predominates. For kinematic excitation, the ratio of the average energies is different.

(a) Analyze the variations in time of both kinds of energy, and of the total energy for the kinematic mode of excitation. Give a reasonable physical explanation for these energy variations. Calculate the ratio of the average values of potential energy to kinetic energy in this case.

(b) At what frequency of the external torque are the average values of the potential and kinetic energy equal to each other?

(c) In the case of dynamic excitation, mean values of both kinds of energy at resonance are equal to one another, and their changes occur exactly in opposite phase, so that total mechanical energy remains constant. However, in the kinematic mode of excitation of forced oscillations, total mechanical energy is subjected to variations even at resonance. Explain these variations. Calculate how much the maximal and minimal values of the total energy differ from its average value (in percent).

1.5 Steady-state Oscillations at Various Frequencies.**

(a) Let the driving frequency ω of the rod be a little less than the natural frequency ω_0 , say, $\omega = 0.9\omega_0$, and let the value of Q be 5. What is the ratio of the amplitude of the steady-state oscillations to the resonant amplitude? What is the phase lag of the oscillation of the flywheel relative to the phase of the rod (in fractions of a cycle)?

(b) At what values of the driving frequency (on either side of the resonant frequency) is the amplitude of steady-state oscillations one half of the resonant amplitude? What is the corresponding phase lag in each of these cases? What kind of energy (averaged over a cycle) predominates in each of these cases?

(c) At what driving frequency ω are the amplitudes of the flywheel and of the rod equal? What is the phase lag of the flywheel relative to the phase of the rod in this case?

1.6 Half-Width of the Resonance Curve.**

(a) Examine analytically and experimentally the dependence of the resonant amplitude of steady-state oscillations on the value of the quality factor Q . How does the half-width of the resonance curve depend on the quality factor Q ? (The *half-width* is the interval of driving frequencies within the limits of which the amplitude of steady-state oscillations exceeds one half of its maximal value.) In other words, how does the sharpness of the resonance peak change when damping is increased?

Make the necessary calculations and then verify your answers by simulating the appropriate experiments on the computer.

(b) How does the position of the resonance peak change as damping is increased? At what value of Q is this maximum shifted to a frequency of zero? (In this case the effect of a static external force applied to the system exceeds that of a driving force oscillating at any frequency.)

1.7 Power Absorbed and Dissipated. Lorentzian.**

(a) Prove analytically that for steady-state forced oscillations the power received by the oscillator from the external source, averaged over a period, equals the averaged value of energy dissipated by friction. Use the values of amplitude a and phase δ for steady-state oscillations, expressed by Eq. (8).

(b) Show that the spectral distribution of power absorbed by the oscillator with weak friction for steady-state oscillations is described by the function

$$F(\omega) = \frac{1}{1 + (\omega - \omega_0)^2 \tau^2},$$

where $\tau = 1/\gamma$. (This function is encountered in various problems of physics. It is called Lorentzian.)

2.2 Transient Processes

2.1* Initial Conditions which Eliminate a Transient.

(a) Under certain initial conditions there is no transient term. That is, immediately after the external driving force is activated, the oscillator executes steady-state oscillations with a constant amplitude at the driving frequency. What are these initial conditions? Express the angle of deflection and the angular velocity corresponding to these initial conditions in terms of the quality factor Q of the oscillator and the parameters of the external action—the driving frequency ω and the amplitude ϕ_0 of oscillations of the rod.

(b) Letting $Q = 5$, $\omega = \omega_0$, and $\phi_0 = 15^\circ$, calculate the initial angle of deflection and the initial angular velocity for which there is no transient. Enter your calculated values and verify experimentally that the forced oscillations are purely sinusoidal from the beginning of the motion. Repeat the same procedure for different values of the system parameters: $Q = 25$, $\omega = 0.5\omega_0$, $\phi_0 = 15^\circ$, and say, $Q = 25$, $\omega = 1.5\omega_0$, $\phi_0 = 15^\circ$.

2.2* Transient Processes at Resonance.

Examine transient processes at resonance experimentally. Enter the following values for the parameters: $Q = 5$, $\omega = \omega_0$, $\phi_0 = 15^\circ$, and enter the initial conditions $\varphi(0) = 0$, $\dot{\varphi}(0) = 0$.

(a) Calculate the lapse of time, measured in units of the period, during which the amplitude reaches 90% of its steady-state value. Verify your answer experimentally. Note the monotonic growth of the amplitude and its exponential asymptotic approach to its steady-state value.

Analyze the character of energy conversions using the graphs of the kinetic, potential, and total energy. At what instants of time is the growth in the total energy of the oscillator a maximum?

(b) Carefully examine the graphs of the decomposition of the resonant transient process into its simple component parts (the transient term and the steady-state oscillation). Note especially the exponential damping of the transient term. Why does the initial value of the amplitude of this natural oscillation equal the amplitude of the steady-state oscillation?

(c)* Taking into account the analytic expression, Eq. (24) for $\varphi(t)$ corresponding to the resonant case ($\omega = \omega_0$), predict the behavior of the Poincaré sections in the phase plane. Verify your prediction by a simulation experiment.

(d)* Consider a transient process at resonance in the absence of friction. How does the amplitude of oscillations increase in time during the transient process which begins from the state of rest in the equilibrium position?

(e)** During a transient process, is it possible for the amplitude to decrease if the frequency of the external force is exactly the resonant frequency? Give physical arguments for your answer. Can you prove your answer experimentally?

2.3* Transient Processes Near Resonance.

Explore transient processes near resonance. Let the frequency of the external force ω be equal, say, $1.2\omega_0$ (ω_0 is the natural frequency of the oscillator). Consider first of all behavior of the system in the absence of friction when the initial conditions are zero.

(a)* Calculate the amplitude of the transient term, that is, the amplitude of oscillations with the natural frequency contributing into the transient process. Also calculate the amplitude of the angular velocity of these oscillations. What is the ratio of the amplitude of the transient term to the amplitude of steady-state forced oscillations? Verify your answers experimentally using the option “Decomposition of the Transient Process”.

(b) Through how many cycles does the rod oscillate before the amplitude of oscillation of the flywheel reaches its maximal value? What is the lapse of time between successive moments at which the amplitude is zero? In other words, what is the beat period?

(c)** Calculate the maximal values of the angular velocity and of the angular displacement during the beats. What is the ratio of the maximal amplitude of the flywheel to the amplitude of the rod?

(d)** Note the distribution of the Poincaré sections of the phase trajectory for this process of beats in the absence of friction. Explain this distribution.

(e)** Consider the effect of friction by entering a moderate value of the quality factor Q (20 – 25), while keeping the previous values of the remaining parameters. What changes in the behavior of the system do you expect? What is the corresponding distribution of the Poincaré sections in the phase plane? Follow behavior of the Poincaré sections for as long time as is needed for the steady-state oscillations to establish. Explain the distribution of the Poincaré sections displayed.

(f)** Change the driving frequency by a small amount. For example, let ω be $1.19\omega_0$ or $1.21\omega_0$ instead of $1.20\omega_0$. What changes in the transient process does this change in the frequency cause? What are the corresponding distributions of the Poincaré sections in the phase plane if there is no friction and if there is weak friction? What are the reasons for the differences in appearance of the display from the case in which $\omega = 1.2\omega_0$?

(g)* Predict the distribution of Poincaré sections for $\omega = 1.25\omega_0$.

2.4* Transient Processes Far from Resonance.

Investigate transient processes far from resonance. Let the frequency of the driving rod be, say, four times smaller than the natural frequency of the oscillator: $\omega = 0.25\omega_0$. Set the two initial conditions to zero.

(a) What are the oscillations like during the transient process if there is no friction? Calculate the amplitudes of the two superimposed oscillations, one with the natural frequency ω_0 and the other with the driving frequency ω . What are the two amplitudes of the corresponding angular velocities?

(b) Observe and explain the shape of the phase trajectory and the distribution of Poincaré sections.

(c) Next introduce moderate friction by setting Q to be approximately 15, keeping the values of the other parameters. Note the gradual fading of the contribution of the natural oscillations. (Use the option “Decomposition of the Transient Process” to see how the time dependencies of these contributing simple oscillations are plotted.) Observe how the complex phase trajectory plotted during the early stage of the transient process is transformed into the ellipse corresponding to steady-state oscillations.

(d)** Consider the opposite case of a driving force with a high frequency. For example, let the driving frequency be, say, four times greater than the natural frequency of the oscillator: $\omega = 4\omega_0$. Set the two initial conditions to zero, and let friction be zero.

What is the ratio of the amplitudes of the two superimposed oscillations with the frequencies ω and ω_0 ? By how many times does the maximal deflection during the transient process exceed the amplitude of steady-state oscillations, contributing into the transient?

(e)** Compare the shapes of the phase trajectories and the distributions of the Poincaré sections for the two cases, $\omega = 0.25\omega_0$ and $\omega = 4\omega_0$, in the absence of friction. How can you explain the similarity in the phase trajectories for these cases?

(f)* Introduce weak friction for the case in which $\omega = 4\omega_0$. Observe the way in which frequent steady-state forced oscillations are established while the contribution of slow transient natural oscillation gradually fades away. How does the phase trajectory evolve in this case? What is the behavior of the Poincaré sections during the corresponding transient process?

(g)** Repeat the simulation experiments for other values of the driving frequency, say $\omega = 0.125\omega_0$, $\omega = 0.5\omega_0$, $\omega = 1.5\omega_0$, $\omega = 2\omega_0$, $\omega = 6\omega_0$). Explain the peculiarities of the transient processes for these cases.

2.3 Supplement: Review of the Principal Formulas

The differential equation of forced oscillations for the kinematic excitation:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = \omega_0^2\phi_0 \sin \omega t.$$

Steady-state forced oscillations are described by the particular periodic solution of this inhomogeneous differential equation:

$$\varphi(t) = a \sin(\omega t + \delta).$$

The amplitude a and the phase shift δ of steady-state forced oscillations:

$$a(\omega) = \frac{\omega_0^2\phi_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}, \quad \tan \delta = -\frac{2\gamma\omega}{\omega_0^2 - \omega^2}.$$

The driving frequency ω_{res} at which the amplitude of steady-state forced oscillations is a maximum is given by:

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0 \left(1 - \frac{\gamma^2}{\omega_0^2}\right) = \omega_0 \left(1 - \frac{1}{4Q^2}\right).$$

The amplitude of steady-state oscillations at resonance (for $\gamma < \sqrt{2}\omega_0$) and its approximate value for $\gamma \ll \omega_0$ are given by:

$$a_{\text{max}} = \frac{\omega_0^2\phi_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}} \approx \frac{\omega_0\phi_0}{2\gamma} = Q\phi_0.$$

Here the quality factor Q is the same dimensionless quantity, $Q = \omega_0/(2\gamma)$, that characterizes the damping of free oscillations.

The amplitude Ω of the angular velocity of steady-state oscillations:

$$\Omega = \omega a(\omega) = \frac{\omega_0^2\phi_0}{\sqrt{(\omega_0^2/\omega - \omega)^2 + 4\gamma^2}}.$$

The maximal amplitude Ω of the angular velocity of steady-state oscillations at resonance (for which $\omega = \omega_0$) is given by:

$$\Omega_{\text{max}} = \omega a(\omega_0) = \frac{\omega_0^2\phi_0}{2\gamma} = \omega_0 Q\phi_0.$$

The initial conditions which eliminate the transient process are:

$$\varphi_0 = a \sin \delta, \quad \dot{\varphi}(0) = a\omega \cos \delta,$$

where a and δ are the amplitude and the phase of steady-state oscillations.

The transient process at the resonant frequency and zero initial conditions:

$$\varphi(t) = -Q\phi_0(1 - e^{-\gamma t}) \cos \omega_0 t.$$