

Forced Oscillations in a Linear System – Problems

Summary of the Principal Formulas

The differential equation of forced oscillations for the kinematic excitation:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = \omega_0^2\phi_0 \sin \omega t.$$

Steady-state forced oscillations are described by the particular periodic solution of this inhomogeneous differential equation:

$$\varphi(t) = a \sin(\omega t + \delta).$$

The amplitude a and the phase shift δ of steady-state forced oscillations:

$$a(\omega) = \frac{\omega_0^2\phi_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}, \quad \tan \delta = -\frac{2\gamma\omega}{\omega_0^2 - \omega^2}.$$

The driving frequency ω_{res} at which the amplitude of steady-state forced oscillations is a maximum is given by:

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0 \left(1 - \frac{\gamma^2}{\omega_0^2}\right) = \omega_0 \left(1 - \frac{1}{4Q^2}\right).$$

The amplitude of steady-state oscillations at resonance (for $\gamma < \sqrt{2}\omega_0$) and its approximate value for $\gamma \ll \omega_0$ are given by:

$$a_{\text{max}} = \frac{\omega_0^2\phi_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}} \approx \frac{\omega_0\phi_0}{2\gamma} = Q\phi_0.$$

Here the quality factor Q is the same dimensionless quantity, $Q = \omega_0/(2\gamma)$, that characterizes the damping of free oscillations.

The amplitude Ω of the angular velocity of steady-state oscillations:

$$\Omega = \omega a(\omega) = \frac{\omega_0^2 \phi_0}{\sqrt{(\omega_0^2/\omega - \omega)^2 + 4\gamma^2}}.$$

The maximal amplitude Ω of the angular velocity of steady-state oscillations at resonance (for which $\omega = \omega_0$) is given by:

$$\Omega_{\max} = \omega a(\omega_0) = \frac{\omega_0^2 \phi_0}{2\gamma} = \omega_0 Q \phi_0.$$

The initial conditions which eliminate the transient process are:

$$\varphi_0 = a \sin \delta, \quad \dot{\varphi}(0) = a\omega \cos \delta,$$

where a and δ are the amplitude and the phase of steady-state oscillations.

The transient process at the resonant frequency and zero initial conditions:

$$\varphi(t) = -Q\phi_0(1 - e^{-\gamma t}) \cos \omega_0 t.$$

1 Steady-state Forced Oscillations

In order to display steady-state oscillations without a preliminary transient process (when the option “Show steady-state” is chosen), the simulation program automatically sets the initial conditions to be $\varphi(0) = a \sin \delta$ and $\dot{\varphi}(0) = a\omega \cos \delta$ (overriding any initial conditions you may have entered). These initial conditions provide steady-state oscillations immediately after the external force is activated. The values of the amplitude a and the phase δ are calculated in the program for the values of the external frequency ω and the quality factor Q you have entered.

1.1 Steady-state Forced Oscillations without Friction.

(a) Setting properties of the system, choose full absence of friction. In this case, the transient process lasts indefinitely, so that steady-state oscillations do not establish. How can you explain the physical sense of the analytical solution that describes steady-state forced oscillations in the absence of friction? Is this solution applicable to a real system? If so, what conditions must be satisfied in order that it were possible in a real system to observe the motion described by this analytical solution?

(b) Convince yourself that for driving frequencies smaller than the natural frequency ($\omega < \omega_0$), the steady-state sinusoidal oscillations of the flywheel occur exactly in phase with the oscillations of the driving rod. At what frequency is the amplitude of the flywheel twice that of the rod? Calculate this frequency and verify your answer with a simulation experiment.

(c) Convince yourself that for driving frequencies greater than the natural frequency ($\omega > \omega_0$), the phase of the steady-state oscillations of the flywheel is exactly opposite that of the exciting rod. At what value of the driving frequency ($\omega > \omega_0$) is the amplitude of the flywheel again twice that of the rod? At what driving frequency are these amplitudes equal? At what frequency is the amplitude of the flywheel one half that of the rod?

1.2* Transformations of Energy for Steady-state Oscillations.

(a) Using the plots of potential, kinetic, and total mechanical energy, find out during which parts of the driving cycle is energy transmitted from the rod to the oscillator. Give a physical explanation of this direction of energy transfer. During which parts of the cycle is energy transferred back from the oscillator to the external source?

(b) For the kinematic excitation of forced oscillations, what is the ratio of the average value of the potential energy to the average value of the kinetic energy for each of those values of frequency for which the amplitude of the flywheel is twice that of the rod? For the case when the amplitudes are equal? For the case when the amplitude of the flywheel equals one half that of the rod? Compare the values observed on the experimental plots with your calculated values.

1.3 The Amplitude and Phase of Steady-state Oscillations.

Examine steady-state oscillations in the presence of friction. Input some moderate value of the quality factor, say, $Q = 5$.

(a) Evaluate the percentage shift of the resonant frequency from the natural frequency.

(b) What is the ratio of the amplitude of steady-state oscillation of the flywheel to the amplitude of the rod at resonance?

(c) What is the phase lag of the oscillation of the flywheel relative to the phase of the rod at the resonant frequency and at a driving frequency equal to 0.8 of the resonant value?

Answer the same questions for $Q = 20$.

1.4** **Peculiarities of the Kinematic Excitation.** In the case of the dynamic excitation of oscillations by a given force whose value is independent of the position of the flywheel, the ratio of the average potential energy to the average kinetic energy equals $(\omega_0/\omega)^2$, so that for low frequencies the potential energy predominates. For kinematic excitation, the ratio of the average energies is different.

(a) Analyze the variations in time of both kinds of energy, and of the total energy for the kinematic mode of excitation. Give a reasonable physical explanation for these energy variations. Calculate the ratio of the average values of potential energy to kinetic energy in this case.

(b) At what frequency of the external torque are the average values of the potential and kinetic energy equal to each other?

(c) In the case of dynamic excitation, mean values of both kinds of energy at resonance are equal to one another, and their changes occur exactly in opposite phase, so that total mechanical energy remains constant. However, in the kinematic mode of excitation of forced oscillations, total mechanical energy is subjected to variations even at resonance. Explain these variations. Calculate how much the maximal and minimal values of the total energy differ from its average value (in percent).

1.5 Steady-state Oscillations at Various Frequencies.**

(a) Let the driving frequency ω of the rod be a little less than the natural frequency ω_0 , say, $\omega = 0.9\omega_0$, and let the value of Q be 5. What is the ratio of the amplitude of the steady-state oscillations to the resonant amplitude? What is the phase lag of the oscillation of the flywheel relative to the phase of the rod (in fractions of a cycle)?

(b) At what values of the driving frequency (on either side of the resonant frequency) is the amplitude of steady-state oscillations one half of the resonant amplitude? What is the corresponding phase lag in each of these cases? What kind of energy (averaged over a cycle) predominates in each of these cases?

(c) At what driving frequency ω are the amplitudes of the flywheel and of the rod equal? What is the phase lag of the flywheel relative to the phase of the rod in this case?

1.6 Half-Width of the Resonance Curve.**

(a) Examine analytically and experimentally the dependence of the resonant amplitude of steady-state oscillations on the value of the quality factor Q . How does the half-width of the resonance curve depend on the quality factor Q ? (The *half-width* is the interval of driving frequencies within the limits of which the amplitude of steady-state oscillations exceeds one half of its maximal value.) In other words, how does the sharpness of the resonance peak change when damping is increased?

Make the necessary calculations and then verify your answers by simulating the appropriate experiments on the computer.

(b) How does the position of the resonance peak change as damping is increased? At what value of Q is this maximum shifted to a frequency of zero? (In this case the effect of a static external force applied to the system exceeds that of a driving force oscillating at any frequency.)

1.7 Power Absorbed and Dissipated. Lorentzian.**

(a) Prove analytically that for steady-state forced oscillations the power received by the oscillator from the external source, averaged over a period, equals the averaged value of energy dissipated by friction.

(b) Show that the spectral distribution of power absorbed by the oscillator with weak friction for steady-state oscillations is described by the function

$$F(\omega) = \frac{1}{1 + (\omega - \omega_0)^2 \tau^2},$$

where $\tau = 1/\gamma$. (This function is encountered in various problems of physics. It is called Lorentzian.)

2 Transient Processes

2.1* Initial Conditions which Eliminate a Transient.

(a) Under certain initial conditions there is no transient term. That is, immediately after the external driving force is activated, the oscillator executes steady-state oscillations with a constant amplitude at the driving frequency. What are these initial conditions? Express the angle of deflection and the angular velocity corresponding to these initial conditions in terms of the quality factor Q of the oscillator and the parameters of the external action—the driving frequency ω and the amplitude ϕ_0 of oscillations of the rod.

(b) Letting $Q = 5$, $\omega = \omega_0$, and $\phi_0 = 15^\circ$, calculate the initial angle of deflection and the initial angular velocity for which there is no transient. Enter your calculated values and verify experimentally that the forced oscillations are purely sinusoidal from the beginning of the motion. Repeat the same procedure for different values of the system parameters: $Q = 25$, $\omega = 0.5\omega_0$, $\phi_0 = 15^\circ$, and say, $Q = 25$, $\omega = 1.5\omega_0$, $\phi_0 = 15^\circ$.

2.2* Transient Processes at Resonance.

Examine transient processes at resonance experimentally. Enter the following values for the parameters: $Q = 5$, $\omega = \omega_0$, $\phi_0 = 15^\circ$, and enter the initial conditions $\varphi(0) = 0$, $\dot{\varphi}(0) = 0$.

(a) Calculate the lapse of time, measured in units of the period, during which the amplitude reaches 90% of its steady-state value. Verify your answer experimentally. Note the monotonic growth of the amplitude and its exponential asymptotic approach to its steady-state value.

Analyze the character of energy conversions using the graphs of the kinetic, potential, and total energy. At what instants of time is the growth in the total energy of the oscillator a maximum?

(b) Carefully examine the graphs of the decomposition of the resonant transient process into its simple component parts (the transient term and the steady-state oscillation). Note especially the exponential damping of the transient term. Why does the initial value of the amplitude of this natural oscillation equal the amplitude of the steady-state oscillation?

(c)* Taking into account the analytic expression for $\varphi(t)$ corresponding to the resonant case ($\omega = \omega_0$), predict the behavior of the Poincaré sections in the phase plane. Verify your prediction by a simulation experiment.

(d)* Consider a transient process at resonance in the absence of friction. How does the amplitude of oscillations increase in time during the transient process which begins from the state of rest in the equilibrium position?

(e)** During a transient process, is it possible for the amplitude to decrease if the frequency of the external force is exactly the resonant frequency? Give physical arguments for your answer. Can you prove your answer experimentally?

2.3* Transient Processes Near Resonance.

Explore transient processes near resonance. Let the frequency of the external force ω be equal, say, $1.2\omega_0$ (ω_0 is the natural frequency of the oscillator). Consider first of all behavior of the system in the absence of friction when the initial conditions are zero.

(a)* Calculate the amplitude of the transient term, that is, the amplitude of oscillations with the natural frequency contributing into the transient process. Also calculate the amplitude of the angular velocity of these oscillations. What is the ratio of the amplitude of the transient term to the amplitude of steady-state forced oscillations? Verify your answers experimentally.

(b) Through how many cycles does the rod oscillate before the amplitude of oscillation of the flywheel reaches its maximal value? What is the lapse of time between successive moments at which the amplitude is zero? In other words, what is the beat period?

(c)** Calculate the maximal values of the angular velocity and of the angular displacement during the beats. What is the ratio of the maximal amplitude of the flywheel to the amplitude of the rod?

(d)** Note the distribution of the Poincaré sections of the phase trajectory for this process of beats in the absence of friction. Explain this distribution.

(e)** Consider the effect of friction by entering a moderate value of the quality factor Q (20 – 25), while keeping the previous values of the remaining parameters. What changes in the behavior of the system do you expect? What is the corresponding distribution of the Poincaré sections in the phase plane? Follow behavior of the Poincaré sections for as long time as is needed for the steady-state oscillations to establish. Explain the distribution of the Poincaré sections displayed.

(f)** Change the driving frequency by a small amount. For example, let ω be $1.19\omega_0$ or $1.21\omega_0$ instead of $1.20\omega_0$. What changes in the transient process does this change in the frequency cause? What are the corresponding distributions of the Poincaré sections in the phase plane if there is no friction and if there is weak friction? What are the reasons for the differences in appearance of the display

from the case in which $\omega = 1.2\omega_0$?

(g)* Predict the distribution of Poincaré sections for $\omega = 1.25\omega_0$.

2.4* **Transient Processes Far from Resonance.**

Investigate transient processes far from resonance. Let the frequency of the driving rod be, say, four times smaller than the natural frequency of the oscillator: $\omega = 0.25\omega_0$. Set the two initial conditions to zero.

(a) What are the oscillations like during the transient process if there is no friction? Calculate the amplitudes of the two superimposed oscillations, one with the natural frequency ω_0 and the other with the driving frequency ω . What are the the two amplitudes of the corresponding angular velocities?

(b) Observe and explain the shape of the phase trajectory and the distribution of Poincaré sections.

(c) Next introduce moderate friction by setting Q to be approximately 15, keeping the values of the other parameters. Note the gradual fading of the contribution of the natural oscillations. (Use the option “Decomposition of the Transient Process” to see how the time dependencies of these contributing simple oscillations are plotted.) Observe how the complex phase trajectory plotted during the early stage of the transient process is transformed into the ellipse corresponding to steady-state oscillations.

(d)** Consider the opposite case of a driving force with a high frequency. For example, let the driving frequency be, say, four times greater than the natural frequency of the oscillator: $\omega = 4\omega_0$. Set the two initial conditions to zero, and let friction be zero.

What is the ratio of the amplitudes of the two superimposed oscillations with the frequencies ω and ω_0 ? By how many times does the maximal deflection during the transient process exceed the amplitude of steady-state oscillations, contributing into the transient?

(e)** Compare the shapes of the phase trajectories and the distributions of the Poincaré sections for the two cases, $\omega = 0.25\omega_0$ and $\omega = 4\omega_0$, in the absence of friction. How can you explain the similarity in the phase trajectories for these cases?

(f)* Introduce weak friction for the case in which $\omega = 4\omega_0$. Observe the way in which frequent steady-state forced oscillations are established while the contribution of slow transient natural oscillation gradually fades away. How does the phase trajectory evolve in this case? What is the behavior of the Poincaré sections during the corresponding transient process?

(g)** Repeat the simulation experiments for other values of the driving frequency, say $\omega = 0.125\omega_0$, $\omega = 0.5\omega_0$, $\omega = 1.5\omega_0$, $\omega = 2\omega_0$, $\omega = 6\omega_0$). Explain the peculiarities of the transient processes for these cases.