Square-wave External Force in a Linear System – Problems

Summary of the Principal Formulas

The differential equation of motion for a torsion spring oscillator driven by a square-wave external force:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = \begin{cases} \omega_0^2\phi_0, & (0,\ T/2), \\ -\omega_0^2\phi_0, & (T/2,\ T). \end{cases}$$

The same equation in which the square-wave shaped right-hand side is represented as a Fourier series:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = \sum_{k=1,\,3,\,5\ldots}^{\infty} \frac{4\phi_0\omega_0^2}{\pi k}\ \sin\omega_k t.$$

The particular periodic solution of the equation (describing steady-state oscillations):

$$\varphi(t) = \sum_{k=1,\,3,\,5\ldots}^{\infty} \frac{4\phi_0}{\pi k}\frac{\omega_0^2}{\sqrt{(\omega_0^2-\omega_k^2)^2+4\gamma^2\omega_k^2}}\ \sin(\omega_k t+\alpha_k),$$

where the phases α_k of the individual harmonics are determined by:

$$\tan\alpha_k = \frac{2\gamma\omega_k}{\omega_k^2-\omega_0^2}.$$

The time dependence of $\varphi(t)$ during the interval $0 \le t \le T/2$, when the equilibrium position is located at $\varphi = \phi_0$:

$$\varphi(t) = \phi_0 + Ae^{-\gamma t}\cos(\omega_1 t+\theta), \qquad (0,\ T/2),$$

where $\omega_1 = \sqrt{\omega_0^2-\gamma^2}$, the frequency of natural damped oscillations, and A and θ are some constants.

The time dependence of $\varphi(t)$ during the interval $T/2 \le t \le T$, when damped natural oscillations occur about the equilibrium position located at $-\phi_0$:

$$\varphi(t) = -\phi_0 - Ae^{-\gamma(t-T/2)}\cos(\omega_1(t - T/2) + \theta), \qquad (T/2,\ T).$$

For a steady-state process, the constants A and θ here have the same values as they do for the interval $(0,\ T/2)$.

1 Swinging of Oscillator at Resonance

1.1 **The Principal Resonance in the Absence of Friction.** Select the idealized case of no friction. Enter the period T of the external force, letting this period be the period T_0 of natural oscillations. Choose null initial conditions. That is, let the oscillator be at rest in the equilibrium position at the moment the external force is activated.

(a) What must be the value ϕ_0 of the angular amplitude of the square-wave motion of the driving rod in order that the amplitude reach 180° after the first 10 cycles? Verify your answer by a simulation experiment.

(b) What regularity does the growth of the amplitude exhibit? Explain the form of the phase trajectory displayed. How does the energy of oscillator grow with time?

(c) If the oscillator is exactly tuned to resonance, is it possible for the amplitude to diminish? Give some physical justification for your answer. Can you test your answer by a simulation experiment?

1.2 **High Resonances in the Absence of Friction.** Examine the resonant excitation of the oscillator initially at rest in the equilibrium position when the period of the external square-wave force is three times longer than the natural period: $T = 3T_0$.

(a) At what value ϕ_0 of the amplitude of a square-wave oscillation of the driving rod does the oscillator reach its maximal deflection of 180° in 10 first cycles of the external action? Verify your prediction experimentally.

(b) What are the differences in the graphs and the phase trajectories between this case and the previous case (Problem 1.1) for which $T = T_0$?

1.3* **Transient Process and Steady-state Oscillations at the Principal Resonance.** Letting the period of the square-wave motion of the driving rod be at the principal resonance, $T = T_0$, and letting the flywheel be initially at rest in the equilibrium position, examine the transient process and the steady-state oscillations in the presence of friction:

(a) Calculate the amplitude of steady-state oscillations for the values $\phi_0 = 10°$ and $Q = 10$. Verify your result experimentally.

(b) What regularity does the growth of the amplitude exhibit in this case? Explain the peculiarities of the phase trajectory.

(c) What is the initial amplitude of damped natural oscillations constituting the transient process for $\phi_0 = 10°$ and $Q = 10$?

(d) What initial conditions cause steady-state forced oscillations to appear from the moment the external force begins to act, thus eliminating the transient process? Verify your answer experimentally.

(e) Examine the spectrum of steady-state oscillations (the output) in this case. Why are these oscillations nearly purely harmonic in spite of the non-sinusoidal, square-wave shape of the input?

1.4* **Steady-state Oscillations at High Resonances.**

(a) Calculate the amplitude of steady-state oscillations for $\phi_0 = 25°$, $Q = 5$, and $T = 3T_0$. Explain the shape of the graphs displayed and of the phase trajectory.

(b) What energy transformations take place during steady-state oscillations? Compare the graphs of the time dependence of the kinetic, potential, and total energy with the corresponding graphs of the angular deflection and the angular velocity of the flywheel. Explain the shape of the graph of the total energy versus the angle of deflection, and explain its relationship to the parabolic potential wells shown in the same diagram.

(c) Which harmonic components determine the shape of output steady-state oscillations in this case? Why, in spite of the exact tuning of the oscillator to the frequency of the third harmonic of the input external force, does the first harmonic component of this force appreciably influence the shape of the output oscillations? How does this harmonic exhibit itself in the pattern of the output oscillations?

(d) Examine the influence of friction on the shape and on the spectral composition of steady-state oscillations at $T = 3T_0$. Note the relative reduction in the contribution of the first and fifth harmonics as the quality factor of the oscillator is increased.

(e) Explore the resonant oscillations of the flywheel when the frequency of the fifth or the seventh harmonic of the external square-wave driving force coincides with the natural frequency of the oscillator. Observe the transformation of the spectrum from input to output, and the dependence of the spectrum of steady-state oscillations on the quality factor of the oscillator. What is the shape of the phase trajectory in these cases? How can you estimate the value of the maximal displacement of the flywheel from the mid-point of its oscillations (from the zero point of the dial) when friction is large (when Q ranges say from 1 to 3)?

2 Non-resonant Forced Oscillations

2.1* **Conditions which Eliminate a Transient at** $T = 2T_0$.

(a) Predict the shape of the graphs of the angular deflection, angular velocity, and the phase trajectory of forced oscillations in the absence friction, for $T = 2T_0$. Under what initial conditions is the steady-state oscillation established immediately after the force is activated? Verify your predictions in a simulation experiment.

(b) Why does the total energy remain constant in these oscillations? When the driving rod executes a jump, why does the oscillator neither gain energy from nor give back energy to the external source?

(c) Examine the spectral composition of steady-state oscillations. Note the contribution of the third harmonic and the influence of its alteration in phase on the shape of the output oscillations: The frequency of the third harmonic in this case is higher than the natural frequency of the oscillator. Therefore its phase in the output oscillations is inverted. As a result, this harmonic component, superimposed on the fundamental, produces a time- dependent graph with bulges at the positions of the flat parts of the square-wave input graph.

2.2 **Steady-state Oscillations for** $T = 2T_0$.

(a) Consider forced oscillations for the case in which $T = 2T_0$ in the presence of friction, by setting $Q \approx 5 - 10$. How do the time-dependent graphs and the phase trajectory differ from the preceding case (Problem 2.1), in which friction is absent? Calculate the maximal deflection of the flywheel attained in these steady-state oscillations. Note changes in the energy transformations.

(b) Why does friction not noticeably influence the spectral composition of steady-state oscillations in this case, in contrast to the case in which $T = 3T_0$?

2.3 **Steady-state Oscillations for a Large External Period.**

(a) Examine forced oscillations for a case in which the natural frequency of the oscillator lies somewhere between the frequencies of two consecutive high odd harmonics of the external action (e.g., let $5T_0 < T < 7T_0$). Which harmonic components dominate in the output steady-state oscillations? Compare the shape of the output steady-state oscillations with the shape of the input square-wave impulses. What is the main difference between the patterns of input and output oscillations?

(b) Investigate the influence of friction on the character of steady-state oscillations. Why are the distortions of the output less prominent the stronger the damping? That is, why is the shape of the output curve for large friction nearly rectangular?

(c) Explain the energy transformations in these oscillations using the graph of the total energy versus the angle of deflection. What is the relationship of this graph with the parabolic potential wells shown in the same diagram?

2.4 **Oscillations Forced by Short-Period Impulses.**

(a) Choose a value for the period T of the square-wave external force to be a small fraction (say $0.2 - 0.3$) of the natural period T_0 of the oscillator. The graph of the angular velocity versus time for steady-state output oscillations has a saw-toothed pattern with teeth which are nearly rectilinear isosceles triangles. Suggest an explanation. What is the difference between the graph of the angular deflection versus time for this case and a sine curve?

(b) Evaluate theoretically the height of a tooth of the angular velocity graph. Evaluate also the maximal deflection angle for these non-sinusoidal steady-state oscillations. Consider the case of weak or moderate friction. Let, for example, the period T be $T_0/4$ and the angle ϕ_0 describing the instantaneous deflections of the driving rod be 30°. What spectral composition is characteristic of such oscillations?