Peculiarities in the energy transfer by waves on strained strings

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Abstract. Localization of elastic potential energy associated with waves in a stretched string is discussed. The influence of nonlinear coupling between transverse and longitudinal waves on the density of energy is investigated by considering the examples of stationary traveling and standing waves. Misunderstandings about different expressions for the density of potential energy encountered in the literature are clarified. The common statement regarding the relationship between the densities of kinetic and potential energies in a transverse wave is criticized.

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1. Introduction

The transport of energy by waves in a string constitutes an important part of the theory of wave propagation. When a wave is present, the string gains both potential and kinetic energies. All the texts and papers agree on the expression for the kinetic energy. However, a serious confusion exists in the literature regarding the density of potential energy associated with waves in a strained string. This confusion originates from the calculation of potential energy stored in a string in a well-known classic text by Morse and Feshbach, Ref. [1]. Comparing two different expressions for the elastic potential energy, Morse and Feshbach came to the conclusion that the potential energy of a string element is not unique. Unfortunately, this erroneous statement spread widely in the literature. Different expressions for the density of elastic potential energy are suggested and discussed in several publications (Refs. [2] – [4]). In a recent contribution to this journal [5] we have tried to clarify these misunderstandings, showing that the elastic potential energy stored in a string depends uniquely on the instantaneous shape of the string. In the present communication we confirm this point of view by examples of stationary periodic traveling and standing waves on a strained string, including cases in which nonlinear coupling between transverse and longitudinal distortions of the string is essential.

2. Energy of transverse waves in a strained string

Our physical model is the standard ideal string which is assumed to be perfectly flexible and linearly elastic. The only restoring force acting on the string elements is a tensile force acting everywhere tangential to the local string direction. Linear elasticity implies that the tensile force is assumed to depend linearly on the amount the string is stretched from its undeformed length. In an undisturbed stretched string each segment already stores some elastic potential energy. However, we are interested here only in the additional potential energy associated with a disturbance caused by the wave. In other words, we assume that the string under tension has zero elastic potential energy in the absence of a wave. For simplicity, we consider planar distortions of the string, which can be described by two scalar quantities: momentary longitudinal displacement $\xi(x,t)$ of a string point whose equilibrium coordinate is x, and displacement $\psi(x,t)$ of this point in the transverse direction. In this section we concentrate on the contribution of transverse distortions $\psi(x,t)$ to the elastic potential energy of the string.

In standard texts that deal with waves in an elastic string (see, for example, Refs. [6], [7]) an assumption is usually made that the slope $\partial\psi/\partial x$ is small everywhere, so that the longitudinal motion of the string points can be neglected in comparison with the transverse motion. In this section we accept this commonly used approximation that all points essentially move only in the transverse direction. Below in Section 3 we consider the possible longitudinal displacements of string points caused by nonlinear effects.

Let us consider an elementary string segment, which in the absence of a wave lies between x and $x+\Delta x$. Its velocity in a transverse wave $\psi(x,t)$ is $\partial\psi/\partial t$. Hence the kinetic energy ΔE_{kin} of the segment and the linear density of kinetic energy $\varepsilon_{\mathrm{kin}}$ (kinetic energy per unit length) are given by the following expressions:

$$\Delta E_{\mathrm{kin}} = \frac{1}{2}\rho_l \left(\frac{\partial\psi(x,t)}{\partial t}\right)^2 \Delta x, \quad \varepsilon_{\mathrm{kin}} = \frac{1}{2}\rho_l \left(\frac{\partial\psi(x,t)}{\partial t}\right)^2, \tag{1}$$

where ρ_l is linear density (mass per unit length) of the stretched string in the absence of a wave.

The calculation of the potential energy associated with the element Δx, or the potential energy density (the potential energy per unit length), is a more subtle matter. Elementary treatments in most textbooks typically assume that additional potential energy ΔE_{pot} of the string element Δx which is disturbed by a transverse wave of a small amplitude, and the corresponding potential energy density $\varepsilon_{\mathrm{pot}}$, are given approximately by the following expressions:

$$\Delta E_{\mathrm{pot}} = \frac{1}{2}T \left(\frac{\partial\psi(x,t)}{\partial x}\right)^2 \Delta x, \quad \varepsilon_{\mathrm{pot}} = \frac{1}{2}T \left(\frac{\partial\psi(x,t)}{\partial x}\right)^2. \tag{2}$$

This expression for ΔE_{pot} is usually treated as the work done by the approximately constant tension T in additional stretching the string element through $\frac{1}{2}(\partial\psi/\partial x)^2\Delta x$ as the element is distorted and displaced transversely from the undisturbed position into the current position with the left end located at $(x, \psi(x,t))$ and the right end — at

$(x+\Delta x, \psi(x+\Delta x, t))$. In [5] we have shown that the approach of Morse and Feshbach [1] for calculation of the elastic potential energy gives the same result, equation (2), if this approach is modified in order to give the true localization of potential energy rather than the energy stored in the whole string.

According to equations (1)–(2), in a purely transverse wave of an arbitrary shape $\psi(x,t) = f(x - v_T t)$ traveling along the string with the speed v_T, the linear densities of kinetic and potential energies are equal to one another at any spatial point x at any time instant t; they rise and fall together. In particular, for a sinusoidal transverse wave $\psi(x,t) = A\sin(k_T x - \omega t)$ (here $k_T = \omega/v_T$), both $\varepsilon_{\rm kin}$ and $\varepsilon_{\rm pot}$ oscillate with frequency 2ω, reaching simultaneously their minimum (zero) values at crests and troughs, and maximum values $\frac{1}{2}\rho_l\omega^2 A^2$ (equal for both) at points of zero displacement $\psi(x,t) = 0$. Clear qualitative descriptions of the energy transformations in a transverse sinusoidal traveling wave can be found in many standard texts on the wave motion. However, the situation may be different when nonlinear effects are taken into account (see Section 3 for details).

For a standing wave described, say, by the wavefunction $A\sin(k_T x)\sin(\omega t)$, the densities of kinetic and potential energies are given by the following expressions:

$$\varepsilon_{\rm kin} = \frac{1}{2}\rho_l A^2\omega^2 \sin^2(k_T x)\cos^2(\omega t), \qquad (3)$$

$$\varepsilon_{\rm pot} = \frac{1}{2}\rho_l A^2\omega^2 \cos^2(k_T x)\sin^2(\omega t). \qquad (4)$$

At the nodes $k_T x = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$, and $\varepsilon_{\rm kin}$ is always zero, while $\varepsilon_{\rm pot}$ oscillates between zero and the maximum value $\frac{1}{2}\rho_l\omega^2 A^2$ with the frequency 2ω. At the antinodes $\cos k_T x = 0$, and $\varepsilon_{\rm pot}$ is always zero, while $\varepsilon_{\rm kin}$ oscillates between zero and the same maximum value $\frac{1}{2}\rho_l\omega^2 A^2$. The density of total mechanical energy in the string $\varepsilon_{\rm kin} + \varepsilon_{\rm pot}$ also oscillates with time with the frequency 2ω and with an amplitude that is position dependent. The energy flow $P(x,t)$ in the standing wave

$$P(x,t) = -T\frac{\partial\psi(x,t)}{\partial x}\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{4}\rho_l\omega^2 A^2 v_T \sin(2k_T x)\sin(2\omega t) \qquad (5)$$

is always zero at nodes and antinodes, where $\sin(2k_T x) = 0$. In particular, the flow equals zero at the end points of the string: the energy of the whole oscillating string is conserved. However, for all points of the string between a node and the adjoining antinodes this is true only for the time average: $\langle P(x,t)\rangle = 0$ over an integer number of half-periods. During a quarter period the energy flow is directed from nodes to antinodes, and during the next quarter period its direction is reversed. Though such quantitative descriptions of energy transformations in a standing wave also can be found in many standard texts, one can encounter in the literature serious misconceptions concerning the energy transfer in a standing wave. For example, the author of Ref. [2] writes: "Unlike the case of a traveling wave, in a standing wave there is no energy transfer, and the total mechanical energy of each string element is expected to be stationary." This is certainly an erroneous statement. Adjoining elements of the string interact and exchange energy even in the standing wave, so that the mechanical energy of an individual string element is not conserved. Indeed, at the moment when the oscillating string passes through its equilibrium, the string energy is wholly kinetic and is localized near the antinodes. Vice versa, a quarter period later the string energy is wholly potential and is localized near the nodes. In the meantime, during this quarter period, the energy travels from antinodes toward nodes transforming simultaneously from the kinetic energy to the potential one. During the next quarter period the string energy is transferred back from nodes to antinodes and transformed simultaneously from the potential energy to the kinetic one.

3. Nonlinear coupling between transverse and longitudinal distortions in a stretched string

Any local stretching of the string caused by a transverse wave in the general case can increase the local tension. One may think that the increased tension, in turn, must excite longitudinal distortions that can take some of the energy of the transverse wave. These small additional longitudinal distortions are produced by transverse waves by virtue of nonlinear effects. Rowland (Refs. [3], [8]) has shown that these nonlinearly induced small longitudinal distortions in a stretched string can give a contribution to the potential energy of the same order of magnitude as the original transverse distortions. This means that generally it may be necessary to take these nonlinear effects into account in calculating the true density of potential energy.

In the approximate nonlinear theory of a linearly elastic perfectly flexible string, in which both transverse and longitudinal motions are taken into account [9], the wave equation for transverse distortion $\psi(x,t)$ is just an ordinary linear wave equation, if terms proportional to the third and higher powers of the distortion are neglected:

$$\frac{\partial^2\psi}{\partial t^2} - v_T^2\frac{\partial^2\psi}{\partial x^2} = 0. \tag{6}$$

In equation (6) $v_T = \sqrt{T/\rho_l}$ is the speed of transverse waves. The wave equation for longitudinal distortion $\xi(x,t)$ in this nonlinear theory [9] includes a forcing term proportional to the second power of the transverse wave amplitude:

$$\frac{\partial^2\xi}{\partial t^2} - v_L^2\frac{\partial^2\xi}{\partial x^2} = (v_L^2 - v_T^2)\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}. \tag{7}$$

In equation (7) the speed of non-forced longitudinal waves is given by $v_L^2 = (SY + T)/\rho_l = (Y/\rho)(1 + T/SY)^2 \approx Y/\rho$, where Y is the Young's modulus, S is the cross-sectional area, ρ is the volume density of the unstrained string material (see, for example, [5]). For an ordinary string (say, piano or guitar string) the tension T is usually small in comparison to SY, which means that its length L in the stretched state under tension T is only slightly greater than its relaxed length L_0 in the absence of tension: $(L - L_0) \ll L_0$.

According to equation (7), nonlinear effects in the general case cause coupling between longitudinal and transverse waves in a taut string. Next we discuss how this coupling influences the spatial distribution of the elastic potential energy associated with a wave in a stretched string.

A slinky spring which is often used as a “string” in lecture demonstrations to illustrate various properties of elastic waves, in relaxed (unstretched) state has a negligible length L_0 compared to its equilibrium length L under constant tension T: $L_0 \ll L$. This means that for a slinky spring $SY \ll T$, and hence transverse and longitudinal waves have almost the same speed ($v_L \approx v_T = \sqrt{T/\rho_l}$). If transverse and longitudinal waves travel with the same speed, equations (6) and (7) decouple, and the two kinds of waves can be perfectly separated. This is the case in which exactly transverse motion is theoretically possible. In a purely transverse wave energy transformations occur just in the way described in Section 2.

However, for all cases when the speed of transverse waves differs from that of longitudinal waves ($v_L \neq v_T$), equation (7) shows that longitudinal waves are necessarily generated whenever transverse waves are excited in a string. This is a nonlinear effect, and the amplitudes of generated longitudinal waves $\xi(x,t)$ that accompany the original transverse wave $\psi(x,t)$, according to equation (7), are proportional to the second power of the original transverse wave amplitude. Nevertheless, as was first indicated by Rowland [8], these longitudinal waves can make a contribution to the density of potential energy in a string of the same order of magnitude as the original transverse wave. Indeed, potential energy of a string segment Δx, associated with longitudinal distortion $\xi(x,t)$ in the strained string, consists of two terms (see, for example, [5]):

$$\Delta E_{\rm pot} = T\left(\frac{\partial\xi}{\partial x}\right)\Delta x + \frac{1}{2}(SY+T)\left(\frac{\partial\xi}{\partial x}\right)^2\Delta x. \tag{8}$$

In the case under consideration the longitudinal distortions are induced by the transverse wave, and, according to equation (7), for a sinusoidal transverse wave of a small amplitude A are proportional to the second power of its amplitude (more exactly, to A^2/λ_T, where $\lambda_T = (2\pi/\omega)v_T$ is the wavelength, see Section 4). Hence in equation (8) only the first term, linear in linear in $(\partial\xi/\partial x)$, is essential. We should take this term into account in order to find the true localization of potential energy. With this contribution of longitudinal distortions, the linear density of elastic potential energy is given by the expression:

$$\varepsilon_{\rm pot} \approx \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2 + T\frac{\partial\xi}{\partial x}. \tag{9}$$

As we have shown in [5], the term proportional to the first power of $(\partial\xi/\partial x)$ in equation (9) for the potential energy density in an infinite sinusoidal wave or in a standing wave on a string with fixed ends does not add any energy to the whole string: this term integrates to zero along the string and hence describes solely some relocation of the potential energy which the string has already stored in the absence of waves by virtue of its preliminary uniform tension. Indeed, in sinusoidal waves the momentary spatial dependence of longitudinal distortion $\xi(x,t)$ is proportional to $\sin(2\pi x/\lambda)$. Hence $(\partial\xi/\partial x) \sim \cos(2\pi x/\lambda)$, and the average value of $T(\partial\xi/\partial x)$ equals zero. Negative values of the term $T(\partial\xi/\partial x)$ appear for the segments of the string which are stretched in the wave by a smaller amount than in the string without a wave, that is, when their elastic energy is smaller than in the preliminary stretched undisturbed string. According to (9), this elastic energy is negative (gives negative contribution to the elastic potential energy of the whole string) due to the arbitrary assumption that the elastic energy of the string without a wave equals zero. If we allow some segment of the preliminary stretched string to quasistatically shorten a bit, the elastic forces exerted by the ends of this segment on the neighbors will do some positive work. This work equals the diminution of elastic energy stored in the segment by virtue of its preliminary stretching. This pre-existing elastic potential energy stored in a segment of the stretched string can be borrowed by its neighbors and then returned in the process of wave motion. This is the physical sense of the term $T(\partial\xi/\partial x)$ in the potential energy density: it describes the spatial redistribution by the wave of the preliminary stored potential energy.

The appearance of the term linear in $(\partial\xi/\partial x)$ in expression (9) for the energy density $\varepsilon_{\rm pot}$ also caused misinterpretations in the literature. According to Rowland [3], "a knowledge of the longitudinal displacement of the mass elements of the string removes the ambiguity in the local potential energy density." We claim that there is no such ambiguity: the knowledge of additional longitudinal distortions caused by a transverse wave simply allows us to definitely determine the redistribution of potential energy in the string.

Next we discuss in detail the relocation of potential energy associated with traveling and standing waves on a strained string by considering certain examples.

4. Relocation of the elastic potential energy in transverse waves caused by induced longitudinal distortions

Let a traveling transverse wave $\psi(x,t) = A\sin(k_T x - \omega t)$ be excited in the string. Then the right-hand side of equation (7) equals $-(v_L^2 - v_T^2)\frac{1}{2}A^2k_T^3\sin 2(k_T x - \omega t)$. Therefore we can search for a partial solution to equation (7) which describes the forced longitudinal motion in the form of a uniform longitudinal wave with the same dependence on x and t, namely $\xi(x,t) = B\sin 2(k_T x - \omega t)$ with $k_T = \omega/v_T$. This induced wave has the frequency 2ω and travels along the string with the same speed v_T as the original transverse wave. Substituting $\xi(x,t)$ in equation (7), we find that such longitudinal wave satisfies equation (7) if its amplitude B is equal to $-\frac{1}{8}k_T A^2$. (The ratio of amplitude B of this wave to the amplitude A of the original wave equals $-\frac{1}{4}\pi A/\lambda_T$.) Hence a possible consistent solution to wave equation (6) for a transverse motion and to wave equation (7) for the coupled longitudinal motion is given by the following expressions:

$$\psi(x,t) = A\sin(k_T x - \omega t), \qquad \xi(x,t) = -\frac{1}{8}k_T A^2 \sin 2(k_T x - \omega t). \qquad (10)$$

Trajectory of a string point in this wave looks like figure eight (Lissajous 2:1 curve). This trajectory is shown schematically (exaggerated) in the left-hand side of figure 1,*a*. The solid curve shows the function $\psi(x,t) = A\sin(k_T x - \omega t)$ at $t = 0$. The displaced circles indicate momentary positions of string points for $t = 0$. The same string points are also shown in their equilibrium positions by circles on the axis.

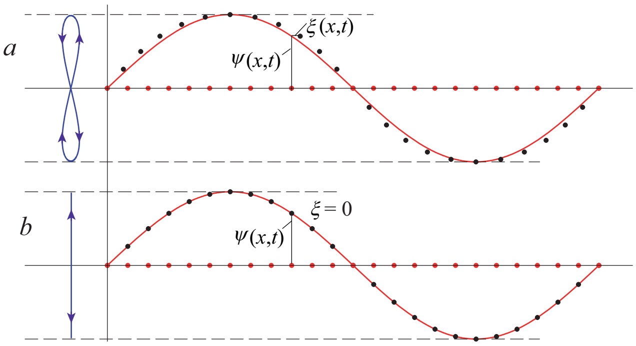

Figure 1. (a) Trajectory of a string point (left) and momentary (for $t = 0$) transverse displacements $\psi(x,t) = A\sin(k_T x - \omega t)$, and longitudinal displacements $\xi(x,t) = B\sin 2(k_T x - \omega t)$ of string points in the traveling wave which is described by equations (10); (b) displacements of the string points in a purely transverse wave.

For comparison, in figure 1,b momentary displacements of the same string points are shown for a purely transverse wave. If nonlinear effects are taken into account, such a wave with $\xi = 0$ can exist only in a “string” like a slinky spring for which transverse and longitudinal waves travel with almost equal velocities ($v_T \approx v_L$), as can be seen from equation (7).

The kinetic energy $\frac{1}{2}\rho_l(\partial\xi/\partial t)^2$ of the induced longitudinal motion is negligible because it is proportional to the square of amplitude B and hence to the fourth power of amplitude A of the transverse motion. Therefore the kinetic energy density $\varepsilon_{\mathrm{kin}}$ in the traveling wave (10), just as in a pure transverse wave, is given by the following expression:

$$\varepsilon_{\mathrm{kin}} = \frac{1}{2}\rho_l\left(\frac{\partial\psi}{\partial t}\right)^2 = \frac{1}{2}A^2 T k_T^2\cos^2(k_T x - \omega t). \tag{11}$$

The most interesting consequence of the nonlinear effect that couples the transverse wave with longitudinal distortions concerns the relocation of elastic potential energy. For the wave with coupled transverse and longitudinal distortions described by equation (10), the potential energy density is given by equation (9). (The omitted term is proportional to the fourth power of amplitude A and hence gives negligible contribution.) Substituting $\psi(x,t)$ and $\xi(x,t)$ from equation (10) in equation (9), we find that the potential energy density $\varepsilon_{\mathrm{pot}}$ in this periodic wave does not depend on time t and coordinate x, that is, the elastic potential energy associated with the wave is uniformly distributed along the string:

$$\varepsilon_{\mathrm{pot}} \approx \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2 + T\frac{\partial\xi}{\partial x} = \frac{1}{4}T k_T^2 A^2 = \pi^2 T\frac{A^2}{\lambda^2}. \tag{12}$$

We compare this remarkable result with the case of a purely transverse wave, in which potential energy density $\varepsilon_{\mathrm{pot}}(x,t)$ at any spatial point at any time instant is equal to kinetic energy density $\varepsilon_{\rm kin}(x,t)$ given by equation (11), and therefore $\varepsilon_{\rm pot}(x,t)$ oscillates at any spatial point between zero and a maximum value $\frac{1}{2}Tk_T^2A^2$ with frequency 2ω. The uniform distribution of potential energy $\varepsilon_{\rm pot}(x,t)$ associated with the periodic wave (10) in a string means that by virtue of the induced longitudinal distortions the string is almost evenly stretched additionally by the wave: we note that in figure 1,a the distances between adjoining circles in the distorted string, being larger than in equilibrium, are equal to one another, in contrast with the unevenly stretched string in a purely transverse wave (figure 1,b).

The instantaneous (for $t = 0$) spatial distributions of kinetic, potential, and total energies in the wave described by equation (10) are shown in figure 2,a. The elastic potential energy associated with this wave is uniformly distributed along the string, and is equal to the average value of potential energy in a purely transverse wave of the same amplitude A, for which $\varepsilon_{\rm pot}(x,t) = \varepsilon_{\rm kin}(x,t)$ (figure 2,e): we emphasize that the induced longitudinal distortions essentially do not add any energy to the wave but cause only a redistribution of the potential energy through the work of the force of tension T which is described by the second term in equation (9). This term is positive for the segments of the string whose length in the wave is increased, and negative for the segments whose length in the wave is diminished. On average, this term (associated with the induced longitudinal distortions) gives no contribution to the potential energy of the wave. The omitted term in equation (9) is proportional to the fourth power of amplitude A and hence gives negligible contribution. Therefore the total amount of elastic potential energy and its average density associated with the combined transverse and forced longitudinal wave described by (10) are determined solely by transverse distortions of the string. In spite of its uniform spatial distribution, this potential energy is transferred by the wave along the string in the direction of propagation with speed v_T together with the kinetic energy. The flow of potential energy through any point is constant, while the flow of kinetic energy oscillates with frequency 2ω. To get an expression for the momentary value of total energy flow $P(x,t)$, which would be consistent with the densities of kinetic energy, equation (1), and potential energy, equation (9), we should include in $P(x,t)$ an additional term $-T(\partial\xi/\partial t)$. This term is equal to the work per unit time done by the force T while the string point is displaced in the longitudinal direction. The time average of this work is zero. This means that on average the energy flow in this wave $\langle P(x,t)\rangle = \frac{1}{2}TA^2\omega k_T = \frac{1}{2}\rho_l\omega^2A^2v_T$ is the same as in the purely transverse wave of amplitude A.

In the above discussion we considered a possible solution, equation (10), of nonlinearly coupled wave equations for transverse and longitudinal distortions. This solution describes an idealized situation in which a sinusoidal transverse traveling wave of infinite length and duration and a forced longitudinal wave whose frequency is twice the frequency of the original transverse wave travel together with the same speed v_T along a uniform infinite string. In such a wave the elastic potential energy is uniformly distributed along the string. The same solution can be applied to a semi-infinite string. Indeed, we can cut the infinite string at some point $x = 0$, forget about its left-hand

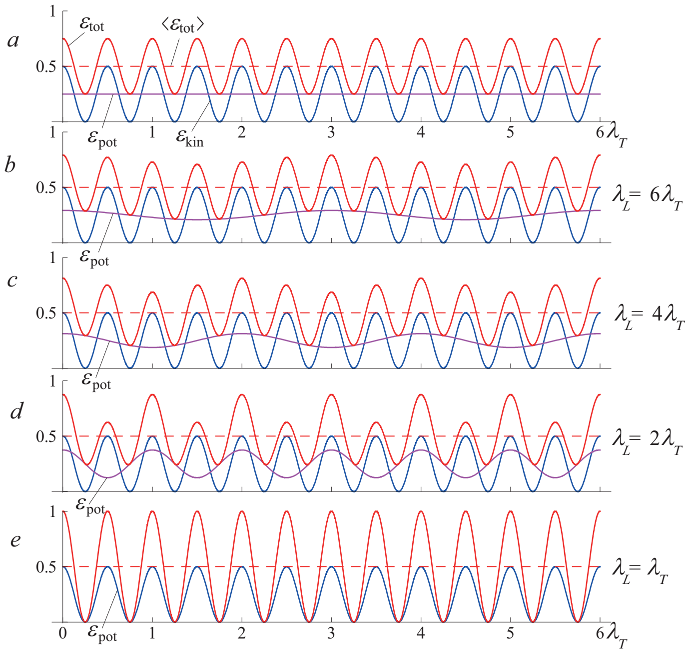

Figure 2. Spatial dependencies of momentary (for $t = 0$) values of the densities of kinetic, potential, and total energy (in units $Tk_T^2A^2$) in the transverse wave $\psi(x,t) = A\sin(k_Tx - \omega t)$ coupled with longitudinal displacements: $\xi(x,t) = B\sin 2(k_Tx - \omega t)$ (a); $\xi(x,t) = B[\sin 2(k_Tx - \omega t) - \sin 2(k_Lx - \omega t)]$ at $v_L = 6v_T$ (b); at $v_L = 4v_T$ (c); at $v_L = 2v_T$ (d); at $v_L = v_T$ (e). Case e corresponds to a purely transverse wave.

semi-infinite part, and force the end point $x = 0$ of the remaining semi-infinite right-hand part to move exactly in the way it moved in the original infinite string, that is, along the figure eight (see figure 1) according to equations

$$\psi(0,t) = -A\sin(\omega t), \qquad \xi(0,t) = \frac{1}{8}\frac{\omega}{v_T}A^2\sin(2\omega t). \tag{13}$$

Then all points between $x = 0$ and infinity of this semi-infinite string will move just in the same way as described by the solution (10). The energy transferred by this wave from $x = 0$ to infinity is supplied by the source that makes the end point $x = 0$ move along the figure eight.

However, it looks more natural to force the left-hand end of the semi-infinite string (the end located at $x = 0$) to oscillate purely transversely, that is, according to $\xi(0,t) = 0$ instead of the figure eight (13). To satisfy this boundary condition at $x = 0$, we should add to solution (10) for the longitudinal distortion a solution of the homogeneous wave equation, namely equation (7) with zero right-hand part. This additional term has the form of a longitudinal wave with frequency 2ω, traveling with the speed $v_L = \omega/k_L$. Therefore the solution to the nonlinearly coupled wave equations for the semi-infinite string whose end $x = 0$ moves transversely has the following form:

$$\begin{aligned}\psi(x,t) &= A\sin(k_Tx - \omega t),\\ \xi(x,t) &= -\frac{1}{8}k_TA^2[\sin 2(k_Tx - \omega t) - \sin 2(k_Lx - \omega t)].\end{aligned} \tag{14}$$

Now with the help of equation (9) we can calculate the density of potential energy in the wave described by the solution (14):

$$\varepsilon_{\rm pot} \approx \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2 + T\frac{\partial\xi}{\partial x} = \frac{1}{4}Tk_T^2A^2\left[1 + \frac{k_L}{k_T}\cos 2(k_Lx - \omega t)\right]. \qquad (15)$$

If the speed of longitudinal waves is much greater than that of transverse waves, that is, if $k_L \ll k_T$, the second term in brackets is negligible. In this case the density of potential energy is almost constant, and, similarly to the solution (10), the potential energy is evenly distributed along the string (figure 2,a). If the speed of longitudinal waves is only several times greater than that of transverse waves, the density of potential energy varies along the string about the same average value $\frac{1}{4}Tk_T^2A^2$ with the spatial period $\lambda_L/2$. Deviations from the mean value the greater the closer v_L to v_T. Graphs of the instantaneous densities for kinetic, potential (equation (15)), and total energy in the wave are shown in figure 2,$b-d$ for different values of $v_L/v_T = \lambda_L/\lambda_T$. If $v_L = v_T$, the longitudinal distortions vanish (according to (14) $\xi(x,t) = 0$), and at any spatial point the momentary density of potential energy equals the density of kinetic energy (figure 2,e), because the wave is purely transverse. In all cases the density of potential energy is well defined, without any ambiguity. We emphasize again that the induced longitudinal distortions cause some relocation of potential energy, but do not add any potential energy to the string: in the above considered stationary traveling waves no energy is transferred from the transverse motion to the induced longitudinal motion.

For a standing transverse wave with the induced longitudinal motion in a taut string, Morse and Ingard, Ref. [9] (see also Refs. [3], [8]), obtained the following solution to nonlinearly coupled wave equations (7):

$$\begin{aligned}\psi(x,t) &= A\sin(k_Tx)\sin(\omega t),\\ \xi(x,t) &= B\sin(2k_Tx)\left(\sin^2(\omega t) - \frac{1}{2}\frac{v_T^2}{v_L^2}\right),\end{aligned} \qquad (16)$$

where the amplitude B of the forced longitudinal wave is

$$B = -\frac{1}{8}A^2k_T = -\frac{\pi}{4}\frac{A^2}{\lambda_T}. \qquad (17)$$

For a string of length L with fixed ends $\lambda_T = 2L/n$ $(n = 1, 2, \ldots)$, $k_T = n\pi/L$, $\omega = k_Tv_T$. The shape of the standing wave described by equations (16) is shown in figure 3,a for an exaggerated amplitude $A = 0.15\lambda_T$. Deviations of this shape from a pure sine curve (which is also shown in figure 3,a) are caused by the induced longitudinal distortions. The string points located at antinodes move purely transversely. Trajectories of points between nodes and antinodes are parabolas (degenerate 2:1 Lissajous curves).‡

According to equation (9), the density of elastic potential energy in the transverse standing wave (16) with the induced longitudinal distortions is given by the following

‡ Trajectories of several string points in the standing wave described by equations (16) are also considered in Ref. [3]. We note that actually these trajectories are parabolas (figure 3,a), but not the inflected curves shown by dashed lines in figure 4 of Ref. [3].

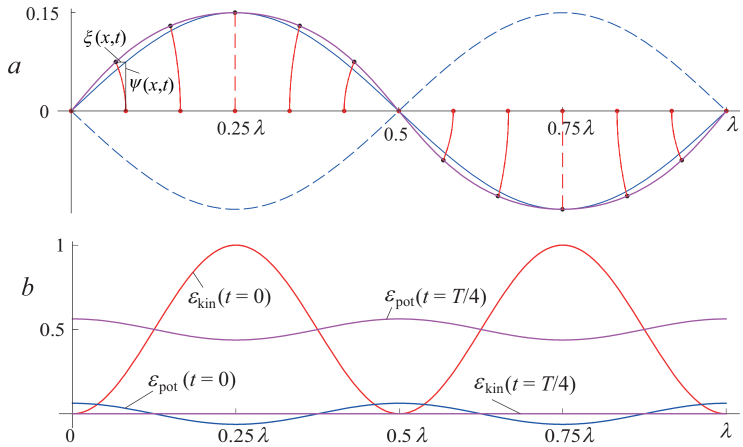

Figure 3. (a) Parabolic trajectories of string points and momentary (for $t = T_0/4$) transverse $\psi(x,t) = A\sin(k_T x)\sin\omega t$ and longitudinal $\xi(x,t) = B\sin(2k_T x)[\sin^2(\omega t) - \frac{1}{2}v_T^2/v_L^2]$ displacements of string points in the standing wave which is described by equations (16); (b) Spatial distribution of kinetic and potential energies (in units $\frac{1}{2}Tk_T^2A^2$) at $t = 0$ and $t = T_0/4$ (quarter period later).

expression:

$$\varepsilon_{\mathrm{pot}} = \frac{1}{4}Tk_T^2A^2\left(\sin^2\omega t + \frac{k_L^2}{2k_T^2}\cos(2k_T x)\right). \qquad (18)$$

For the moment $t = T_0/4$ ($T_0 = 2\pi/\omega$) all the string points reach their utmost displacements (see figure 3,a). Kinetic energy at this moment is zero, and potential energy reaches its maximum. If $v_L/v_T = \lambda_L/\lambda_T \gg 1$, potential energy is distributed almost uniformly along the string; its density $\varepsilon_{\mathrm{pot}}$ is slightly greater than the mean value near the nodes, and $\varepsilon_{\mathrm{pot}}$ is slightly smaller near the antinodes (figure 3,b). During a quarter period all potential energy is transformed into kinetic energy, which is localized near the antinodes — see graph of $\varepsilon_{\mathrm{kin}}$ for $t = 0$. We note that at time moment $t = 0$ when the total energy of the string is equal to kinetic energy and potential energy is zero, the density of potential energy is not zero: $\varepsilon_{\mathrm{pot}}$ is positive near the nodes and negative near the antinodes. This means that at $t = 0$ when the string passes through equilibrium position, its tension is not exactly uniform. Indeed, points of the string between a node and antinode move along slightly curved trajectories (figure 3,a). To provide these points (elementary segments of the string) with necessary centripetal acceleration, the elastic forces exerted on the string segment by its neighbors must give a resultant force directed towards the center of curvature of the trajectory.

In the limiting case $v_L \to v_T$, which corresponds to the slinky spring, the nonlinear coupling term in the right-hand side of equation (7) for the longitudinal distortion vanishes. This means that at $v_L = v_T$ a purely transverse standing wave is possible. However, a nonzero induced longitudinal motion holds in the limit $v_L \to v_T$ in the solution described by equation (16). Such coexistence of several different stationary solutions at the same values of all parameters is a manifestation of multistability — a characteristic feature of nonlinear systems.

5. Concluding remarks

In this paper we concentrated on the localization of elastic potential energy associated with waves on a string. Considering examples of stationary sinusoidal waves, we have tried to clarify misunderstandings and contradictions encountered in the literature regarding the energy of waves on strained strings. We emphasize that there is no inherent ambiguity in elastic potential energy associated with transverse waves, contrary to the widespread opinion originating from the classic textbook of Morse and Feshbach, Ref. [1]. When the transverse and longitudinal distortions of the string caused by the wave are known, the localization of potential energy is uniquely defined by equations (2) and (8) for the potential energy stored in an individual segment of the string. Due to the nonlinear coupling, a transverse wave can generate in the string small longitudinal distortions. Occurrence in equation (9) of the term proportional to the first power of the longitudinal distortion explains the redistribution by the wave of the potential energy, already stored in the string by virtue of its preliminary stretching. This unambiguous relocation of the elastic potential energy is illustrated in the paper by considering the examples of sinusoidal traveling and standing nonlinearly coupled transverse and longitudinal waves. The average energy carried by the combined transverse and nonlinearly induced longitudinal traveling wave is equal to that of the original transverse wave. The only change caused by the nonlinear coupling is revealed in the spatial distribution of potential energy due to the additional longitudinal distortions generated by the wave as it moves in the strained string.

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