The Physics of the Oceanic Tides

Eugene I. Butikov
St. Petersburg State University, St. Petersburg, Russia

Abstract

Common misconceptions about tides encountered in the literature are briefly discussed. A physically transparent simple but rigorous derivation of tide-generating forces is suggested, followed by a dynamical treatment of the tidal wave in a simplified model of the ocean as a water shell of equal depth wholly covering the globe. Novelty of our approach consists in representing tide-generating forces as two quadrupole systems of oscillating forces with axes of symmetry making 45° angle with one another. The dynamical response of the ocean to these time-dependent forces is found as steady-state forced oscillations in a linear system. The treatment is appropriate for scientists in oceanography, geosciences and astronomy, and also for a wide readership, including non-specialists and enthusiasts interested in the phenomenon. A set of simulations is developed to aid understanding the phenomenon of tides, which is of great general interest due to its cosmic nature.

Keywords: tide-generating forces, reference frames, tidal bulges, forced oscillations, tidal wave, phase lag.

1 Common misconceptions about the tides

Numerous treatments of tides encountered in some textbooks and web sites are either plainly wrong or at least misleading. The authors who write about tides usually don't get into the physics theory much. Comprehensive discussions of typical textbook and media mistakes are presented on the web by Donald Simanek (2015). Various misleading representations of tides in press and media are discussed by Mikolaj Sawicki (1999). Some of the misconceptions are the result of their author's misunderstandings. However, the most evident common reason of misconceptions in textbooks on oceanography, such as Thurman (1994) or Ross (1995), is probably related with inaccurate or misused terminology in describing the origin of the tide-generating forces.

Much confusion regarding the origin of tides arises from the incorrect choice of the reference frame. When a non-inertial reference frame is used for the explanation, the pseudo force of inertia, which is responsible (together with gravitational forces) for the origin of tides on the earth, is caused by the *translational* acceleration of the earth relative to the inertial space. This acceleration is just the "free-fall acceleration" of the earth under the gravitational pull exerted on the earth as a whole by the celestial body that causes the tides. The force of inertia associated with this translational acceleration has the same magnitude and direction everywhere on the earth and in its vicinity. Some authors, when writing on tides, call this force of inertia (caused by the translational acceleration of the earth) the "centrifugal force" (see, for example, Sirtoli, 2005, NOOA, 2016). This terminology is misleading, because the "centrifugal" and "centripetal" language is usually associated with the rotational motion. However, the earth's orbital revolution around the sun (as well as the "whirling" of the earth-moon system around its center of mass) is a "revolution without rotation" (essentially a *translational* motion) of the relevant geocentric frame of reference. To avoid misunderstandings, we shouldn't call this force of inertia, caused by a translational acceleration of the reference frame, a "centrifugal force."

It makes sense to divide the problem of tides into two parts: the first part concerns the origin and properties of tide-generating forces, and the second one relates to the much more complicated issue of the dynamical effect that these time-varying forces have on the ocean. We note that much of the confusion in the literature is related already to the first (rather simple) part of this problem, which can be completely and unambiguously solved using Newtonian mechanics. We have already tried to clarify the issue in an earlier publication on the subject (Butikov, 2002). The present paper is a further development of this approach.

2 Explanation of tidal forces in terms of physics

In this paper we present a simple but rigorous physically transparent derivation of the moon- and sun-induced tide-generating forces in the geocentric reference frame. Then we consider a dynamical response of the ocean to these forces. For this second part, we restrict our analysis to a simplified model of the ocean as a uniform water shell of equal depth wholly covering the globe. We show how this problem can be reduced to the well-known behavior of a linear damped oscillator under sinusoidal forcing.

We emphasize that our simplified treatment in the present paper is intended only to make clear the physical background of the tides and does not describe the complete picture. The purely theoretical quantitative description of tides for a given location on the earth, derived solely from first principles, is hardly possible because of the extremely complex structure of the oceans – the actual dynamical system that responds with tides and tidal currents to the well known tide-generating forces.

Next we discuss qualitatively the physical nature of the sun- and moon-induced tide-generating forces in a non-rotating geocentric frame of reference. We determine the static (equilibrium) distortion of the ocean surface under these forces. Then we obtain expressions for the tidal forces that are applicable on the rotating earth, and discuss how these forces depend on time by virtue of the daily axial rotation of the earth.

We show mathematically and illustrate by a simulation program that a uniform rotation of the system of tidal forces, which are coupled with the apparent motion of the sun (or moon), can be represented as a superposition of two oscillating quadrupole systems of non-rotating forces whose axes make an angle of 45° with one another. Each of these systems of forces generates a steady-state forced oscillation of the ocean (a standing wave). In this way the problem of dynamical response of the ocean to time-dependent tidal forces can be reduced to the well-known problem of forced oscillations in a linear system. We can treat the tidal wave circulating around the globe in the adopted simplified model as a superposition of the two standing waves. Finally the real-world complications of this simplified picture are discussed briefly, as well as the role of tidal friction in the evolution of the axial rotations and orbital revolutions of celestial bodies.

The paper is accompanied by a set of specially developed simulations implemented as Java applets that run directly in the browser (Butikov, 2012). These simulations aid understanding various aspects of this rather difficult but interesting and important issue concerning the origin and properties of tides. By using the simulations, it is possible to visualize some non-trivial aspects of this natural phenomenon which is of interest for many people due to its cosmic nature.

2.1 Elementary explanation of the origin of tidal forces

Our planet earth moves with an acceleration relative to the heliocentric inertial reference frame. This acceleration is produced by the gravitational attraction of the earth to the sun and also to the moon and to all other celestial bodies. Actually, this is just the acceleration of free fall experienced by the earth as a whole in the resultant gravitational field produced by all the nearby and far-off celestial bodies. We emphasize that this acceleration of the earth is independent of the velocity of its orbital motion around the sun. This acceleration is also independent of the velocity which the earth has by virtue of its whirling with the moon around their common center of mass. The earth would move with the same acceleration were it freely falling in this external gravitational field. This acceleration is also independent of the daily rotation (spin) of the earth. What is actually important for the physical explanation of tide-generating forces, is only the acceleration of the earth, not its orbital velocity and daily rotation.

Hence, to better understand the origin of tide-generating forces, we first use a non-rotating geocentric reference frame. The origin of this frame is at the center of the earth, and the axes are directed toward immovable stars: This frame is not involved in the daily rotation of the earth. Although the origin of this frame moves approximately in a circle around the sun (more exactly, around the center of mass of the sun-earth system), the frame itself does not rotate because the directions of its axes are fixed relative to the distant stars. The motion of this frame—revolution without rotation—is a *translational* (though nearly circular) motion. We can figuratively compare the motion of this frame with the circular motion of the frying pan in the hands of a cook while the handle of the pan points in the same direction.

With respect to the inertial space, all points of this reference frame move with an acceleration $\mathbf{a}_0$ whose magnitude and direction are the same for all the points of this frame. Any body of mass m whose motion is referred to this non-inertial geocentric frame (for example, an earth's satellite, or a drop of water in the ocean) is subject to the pseudo force of inertia, $\mathbf{F}_{\text{in}} = -m\mathbf{a}_0$. This force is independent of the position of the body relative to the earth. In other words, the field of this force of inertia is uniform. At the same time, the net gravitational field of the sun, moon and all other celestial bodies, which creates the acceleration $\mathbf{a}_0$ of the earth, is non-uniform. It is this non-uniformness of the external gravitational field that is responsible for the origin of tidal forces.

Regarding the role of the daily axial rotation (spin) of the earth in the explanation of tides, we note the following. Due to this daily rotation (and to a lesser extent due to the orbital motion of the moon), the apparent positions of celestial bodies, which create the tidal forces, are changing with time. Thus, these movements are responsible only for the character of time-dependence of tidal forces in a particular place on the rotating earth. We will take this time-dependence into account later on in our analysis of the influence of tidal forces on the oceanic waters.

We emphasize that the uniform force of inertia, $\mathbf{F}_{\rm in} = -m\mathbf{a}_0$, which is responsible for the tides (together with the gravitational force), must be introduced in the non-inertial frame of reference whose motion is *translational* (though almost circular), but not a *rotational* motion. Explaining the tides, the authors of textbooks and papers often use the term "centrifugal" to denote this pseudo force of inertia (see, for example, NOAA, 2016, Sirtoli, 2005). Probably this causes confusion and misconceptions in the literature, because the term "centrifugal" is usually associated with rotation.

Next for simplicity we explain the origin of sun-induced tidal forces. Tidal forces produced by the moon (or any other celestial body) can be explained quite similarly. There is no essential difference in the physics of sun-induced or moon-induced tidal forces. The tidal forces experienced by a body on the earth, which are produced by several external sources (predominantly by the moon and sun), simply superimpose (add vectorially).

Hence further on let $\mathbf{a}_0$ be the acceleration of the earth produced only by the gravitation of the sun. If the body were placed at the center of the earth, the pseudo force of inertia $\mathbf{F}_{\rm in} = -m\mathbf{a}_0$ would exactly balance (cancel) the gravitational attraction $\mathbf{F}_{\rm gr} = m\mathbf{a}_0$ of the body to the sun. In other words, if we consider the earth as a giant spaceship orbiting the sun, a body placed at the center of this ship would seem to be weightless with respect to the gravitation of the sun, just as astronauts on an orbital station seem to be weightless in the gravitational field of the earth.

The pseudo force of inertia $\mathbf{F}_{\rm in} = -m\mathbf{a}_0$ experienced by a body in the geocentric frame (in the frame that revolves about the sun-earth center of mass without axial rotation) has the same magnitude and direction everywhere on the earth—it is independent of the coordinates. On the other hand, the gravitational pull of the sun $\mathbf{F}_{\rm gr}$ experienced by the same body diminishes with its distance from the sun and is directed to the center of the sun, and hence both the magnitude and direction of $\mathbf{F}_{\rm gr}$ depend on the position of the body on the earth. It is the combined action of $\mathbf{F}_{\rm gr}$ and the force of inertia $\mathbf{F}_{\rm in}$ that we call the sun-induced *tidal force* (or tide-generating force).

These differential effects of gravity described by tidal forces give rise, in particular, to solar (and lunar) gravitational perturbations of an earth satellite's geocentric orbit. The tide-generating forces slightly distort the earth's gravitational pull that governs the satellite's motion. This means that the satellite, after a revolution, does not return exactly to the same point of the geocentric reference frame. On the surface of the earth, these same forces give rise to the tides. We emphasize again that tidal forces are caused not by the sun's (or moon's) gravitational field itself, but rather by the non-uniformity of this field.

Describing the tides, we can avoid using a non-inertial reference frame, being not inclined to introduce the concept of the pseudo force of inertia. In doing so, we can use a somewhat different language in the derivation of the tidal force: Instead of discussing the vector addition of the pull of the sun and the corresponding pseudo force of inertia arising from the non-inertial character of the geocentric reference frame, we can use instead an inertial (heliocentric) frame. In this approach the tidal force experienced by an earthly body can be found by the vector subtraction of the gravitational force of the sun on the body at its given location with the gravitational force of the sun on the same body were it located at the center of the earth. The two ways of thinking about it are equivalent and yield the same results.

Indeed, when viewing the situation on the earth from the inertial frame of reference, we can apply the Galilean law according to which, in the same gravitational field (here the field of the sun), all free bodies experience equal accelerations. Hence the earth as a whole and all free bodies on the earth, being subjected to almost the same solar gravitational field, are very nearly accelerated toward the sun. Consequently we do not particularly notice the influence of solar gravitation on what happens on the earth. The small differences between the acceleration of the earth as a whole and of the earthly bodies depend on the distances of the bodies from the center of the earth, because these differences are caused by the non-uniformity of the solar gravitational field over the extent of the earth. Although such approach allows us avoid introducing pseudo forces, the usage of a non-inertial frame of reference that moves translationally is physically more transparent and makes the mathematics simpler.

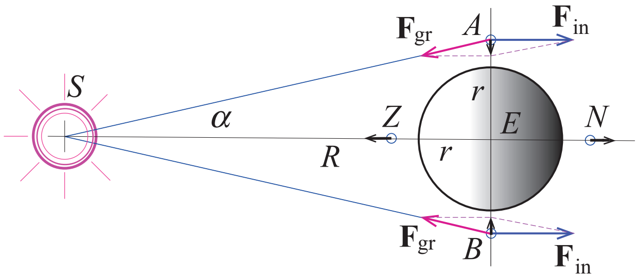

Figure 1: The tidal forces at different points over the earth.

Figure 1 illustrates the origin and properties of the tide-generating forces produced by the sun on the earth and in its vicinity. On this schematic image, the size of the earth E is strongly exaggerated compared to the earth-sun distance R. As for the spatial size of the sun, it doesn't matter at all, because in this problem the sun plays only a role of the source of the gravitational force, and consequently all its mass can be considered as if concentrated in its center.

The free-fall acceleration of the earth E in the gravitational field of the sun S (relative to the inertial space) is $a_0 = GM_{\text{sun}}/R^2$, where M_{sun} is the mass of the sun, and R is the sun-earth distance. We note again that only the acceleration caused by the gravitation of the sun is important, but not the velocity of the earth in its orbital motion. The gravitational pull of the sun $\mathbf{F}_{\text{gr}}$ experienced by any body (e.g., by a satellite) at point A almost equals the pseudo force of inertia $\mathbf{F}_{\text{in}}$ in magnitude, because the distances of the body and of the center of the earth from the sun are almost equal. However, the gravitational force here is directed not exactly opposite to the inertial force. Thus, their non-zero resultant, the tidal force $\mathbf{F}_A$, is directed toward the earth (vertically downward). Figure 1 allows us to evaluate its magnitude:

$$F_A \approx F_{\text{gr}}\alpha \approx F_{\text{in}}\alpha \approx ma_0\frac{r}{R}, \tag{1}$$

where $\alpha = r/R$ is the angular distance between the body and the center of the earth as seen from the sun. The tidal force $\mathbf{F}_B$ at the opposite point B equals $\mathbf{F}_A$ in magnitude and is also directed vertically downward to the earth. On the surface of the earth, the tidal force is directed vertically downward at all places (forming a circle) where the sun is in the horizon at that moment.

The distance from the sun to the body at point Z (for which the sun is at the zenith) is smaller than to the center of the earth. Here the gravitational pull of the sun is somewhat greater than the pseudo force of inertia. Hence, the tidal force $\mathbf{F}_Z$ at this point is directed vertically upward (from the earth toward the sun). Its magnitude

$$F_Z = G\frac{mM_{\rm sun}}{(R-r)^2} - ma_0 = ma_0\left[\frac{R^2}{(R-r)^2} - 1\right] \approx ma_0\frac{2r}{R} \tag{2}$$

is approximately twice the magnitude of the tidal forces at points A and B. Similarly, at the opposite point N (for which the sun is at nadir) the pseudo force of inertia is greater than the gravitational pull of the sun, and so the tidal force $\mathbf{F}_N$ is also directed vertically upward from the earth (and from the sun). In magnitude $\mathbf{F}_N$ approximately equals $\mathbf{F}_Z$.

We note again that the tidal forces depend on the acceleration a_0 of the earth. They are independent of the orbital motion of the earth and would also occur if the earth and sun were simply falling toward each other under the mutual gravitation, or moving in the inertial space with arbitrary velocities.

The expressions for the tidal forces, $F_A = ma_0r/R = (GmM_{\rm sun}/R^2)(r/R)$, Eq. (1), and $F_Z \approx 2F_A$ given by Eq. (2), are valid also for the tidal forces produced on the earth by the moon if we replace $M_{\rm sun}$ by the mass of the moon $M_{\rm moon}$ and R by the moon-earth distance. There is no intrinsic difference between the physical origin of sun-induced and moon-induced tide-generating forces. In both cases, the only important factor is the acceleration of the earth under the gravitational pull of the celestial body that causes the tides on the earth, not the orbital velocities of both gravitationally coupled bodies (the earth and the sun, or the earth and the moon).

The tidal force experienced by any object is proportional to its distance r from the center of the earth and inversely proportional to the cube of the distance R to the celestial body that causes the force, and is proportional to the mass of the source body. As noted, lunar tide-generating forces on the earth are more than twice those of the sun (their ratio is approximately 2.2) because the moon is much closer to the earth.

2.2 Horizontal and vertical components of the tidal force

The sun-induced tide-generating forces exerted on the earthly bodies and on the earth itself have a quadrupole character: they stretch the earth along the sun-earth line, and squeeze the earth in the directions perpendicular to that line, as shown in Figure 2. At points A and B tidal forces are directed vertically downward; at points Z and N – vertically upward. At these four points tidal forces does not have horizontal components. At intermediate points tidal forces have intermediate directions, and have non-zero horizontal components (see Figure 2). The system of tidal forces has an axial symmetry with respect to the sun-earth line.

Calculation of tidal forces for arbitrary points requires a somewhat more sophisticated mathematics (see Appendix I). Because of the axial symmetry with respect to the sun-earth line, the vertical and horizontal components of the tidal force at all points (located at the same distance r from the center of the earth) depend only on the angle that determines the position of the mass point m on or near the surface of the earth measured from the direction to the sun (this angle θ is shown in Figure 3).

In Figure 3, vector $\mathbf{R}$ is the radius vector of the earth relative to the sun, and $\mathbf{r}$ is the radius vector of some mass point m relative to the center of the earth. Vectorial addition of the uniform force of inertia and the force of gravitational attraction of the mass point m

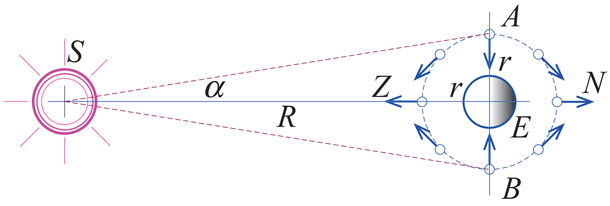

Figure 2: Quadrupole configuration of tidal forces at different points over the earth's surface.

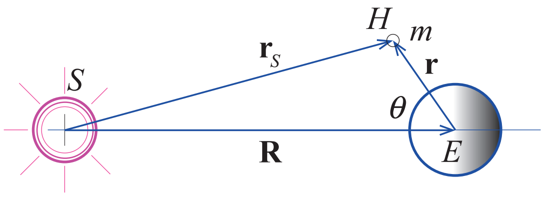

Figure 3: For the calculation of the tidal force at an arbitrary point $H(r,\,\theta)$ near the earth.

to the sun yields for the tidal force $\mathbf{F}_{\text{tid}}$ at an arbitrary position the following expression (for the relevant calculation see Appendix I, Eq. (14)):

$$\mathbf{F}_{\text{tid}} \approx -G\frac{mM_{\text{sun}}}{R^3}\left[\mathbf{r} - 3\mathbf{R}\frac{(\mathbf{R}\cdot\mathbf{r})}{R^2}\right]. \qquad (3)$$

Multiplying the right-hand side of Eq. (3) by the unit vector $\mathbf{r}/r$ directed from the center of the earth to the point in question, we obtain the following dependence of the vertical component of the tidal force $\mathbf{F}_{\text{tid}}$ on the angle θ (see Figure 3) between the directions of vectors $\mathbf{r}$ and $(-\mathbf{R})$:

$$(F_{\text{tid}})_{\text{vert}} = G\frac{mM_{\text{sun}}}{R^3}\,r(3\cos^2\theta - 1) = F_{\text{gr}}\frac{r}{R}(3\cos^2\theta - 1). \qquad (4)$$

We remind that θ is the angle between the directions to the point in question and to the celestial body that is the source of the tidal force (see Figure 3). At all the points for which $\cos\theta = \pm 1/\sqrt{3}$, $(F_{\text{tid}})_{\text{vert}} = 0$, that is, the tidal force is directed horizontally.

For the horizontal component of the tidal force at an arbitrary point H whose geocentric position is determined by coordinates r and θ (see Figure 3), Eq. (3) yields:

$$(F_{\text{tid}})_{\text{hor}} = -3G\frac{mM_{\text{sun}}}{R^3}\,r\,\cos\theta\sin\theta = -\frac{3}{2}F_{\text{gr}}\frac{r}{R}\sin(2\theta). \qquad (5)$$

The horizontal component $(F_{\text{tid}})_{\text{hor}}$ is a maximum at all the points of the earth for which $\theta = \pm 45^\circ$ and $\theta = \pm 135^\circ$. This maximum value of the horizontal component equals $(3/2)(r/R)F_{\text{gr}}$ or, equivalently, $(3/2)(r/R)GmM_{\text{sun}}/R^2$. The horizontal component of the solar tide-generating force causes a deviation of the plumb line from the direction of the earth's own gravity (at $\theta = \pm 45^\circ$) only by $0.008''$.

The magnitude of the tidal force experienced by a water drop in the ocean or by an artificial satellite is proportional to its distance r from the center of the earth and inversely proportional to the *cube* of the distance R to the celestial body that causes the force.

Expression (4) for the vertical component of the tidal force $\mathbf{F}_{\text{tid}}$ can be transformed to the following equivalent form:

$$(F_{\text{tid}})_{\text{vert}} = G\frac{mM_{\text{sun}}}{R^3}\,r(3\cos^2\theta - 1) = \frac{3}{2}G\frac{mM_{\text{sun}}}{R^2}\,\frac{r}{R}(\cos 2\theta + \frac{1}{3}). \tag{6}$$

The last term in the right-hand expression of Eq. (6) is independent of θ. Hence it is the same everywhere on the earth (for a given value of r) and adds only a tiny constant value to the vertical force of the earth's gravity mg (on the earth's surface about ten million times smaller than mg). This term in the vertical component of the tidal force is thus independent of time on the spinning earth. It can therefore be dropped as far as the tides are concerned.

Thus, the vertical and horizontal components of the tidal force exerted on a body of mass m located near the earth at a position determined by angle θ and radius r are given by the following expressions:

$$(F_{\text{tid}})_{\text{vert}} = \frac{3}{2}F_{\text{gr}}\frac{r}{R}\cos(2\theta); \quad (F_{\text{tid}})_{\text{hor}} = -\frac{3}{2}F_{\text{gr}}\frac{r}{R}\sin(2\theta), \tag{7}$$

where $F_{\text{gr}} = GmM_{\text{sun}}/R^2$ is the total gravitational pull of the sun (or the moon) experienced by the body anywhere on the earth. This representation of the tide-generating force is especially convenient because Eqs. (7) define a tidal force vector whose magnitude $(3/2)(r/R)F_{\text{gr}} = (3/2)GmM_{\text{sun}}/R^2(r/R)$ is independent of the angle θ: the tidal forces at all points that lie at a given distance r from the earth's center are equal in magnitude and differ only in direction. Hence it is admissible to consider that all the vectors of tidal forces shown in Figure 2 for different points at the same height over the earth have the same length.

Certainly, the system of the moon-induced tidal forces also has a quadrupole character. Equations (7) are also valid for the tidal forces produced by the moon, provided we replace the mass of the sun M_{sun} by the mass of the moon M_{moon} and the sun-earth distance R by the moon-earth distance. In this case the angle θ in Eqs. (7) determines position of the body relative to the earth-moon line. The entire pattern of tidal forces is tied to the sun's or moon's position and follows its apparent path around the earth.

The lunar tide-generating force, $F_{\text{tid}} = (3/2)GmM_{\text{moon}}r_0/R^3$, experienced by a body of mass m on the surface of the earth (here r_0 is the earth's radius), is very small compared to its weight, that is, to the earth's force of gravity $F_{\text{grav}} = mg = GmM_{\text{earth}}/r_0^2$. If we let the ratio $M_{\text{moon}}/M_{\text{earth}} = 1/81$ and the mean earth-moon distance $R = 60\,r_0$ (actually this distance varies between $57\,r_0$ and $63.7\,r_0$ because of the elliptical shape of the moon's orbit), we obtain:

$$\frac{F_{\text{tid}}}{F_{\text{grav}}} = \frac{F_{\text{tid}}}{mg} = \frac{3}{2}\frac{M_{\text{moon}}}{M_{\text{earth}}}\frac{r_0^3}{R^3} = 8.6 \cdot 10^{-8}. \tag{8}$$

Although the tide-generating forces are very small in comparison with the earth's force of gravity (on the surface of the earth the lunar tidal force at its maximum being only about 10^{-7} times the earth's gravitational force), their effects upon the ocean water are considerable because of their horizontal component, which is orthogonal to the earth's gravitational field and varies with time periodically because of the earth's axial rotation. This horizontal component shifts the ocean water around the globe. We emphasize again that the horizontal (tangential to the surface) components of the tidal forces are much more influential on the ocean tides and on the orbits of earth's artificial satellites compared to the vertical (radial) components of the tidal forces, which only modify slightly the earth's gravitational force.

2.3 The static distortion of the water surface

To estimate the static (equilibrium) distortion of the ocean's surface due to the tidal forces, we can use the hypothetical situation of a non-rotating planet on which the tide-generating forces are nearly time-independent. (This model would be strictly applicable to a planet whose axial rotation is synchronized with the orbital motion of its moon that causes the tidal forces.) From the symmetry of tidal forces, Eqs. (7), illustrated also in Figure 2, we can assume that the distorted surface with the "tidal bulges" at points Z and N has an ellipsoidal shape of the same symmetry. Mathematically, this spheroid (which is characterized by an axial symmetry with respect to the sun-earth line) is given by the following expression:

$$r(\theta) = r_0 + a\cos 2\theta, \tag{9}$$

where $2a \ll r_0$ is the difference between the static maximal and minimal levels at points Z and A, respectively (see Figure 4).

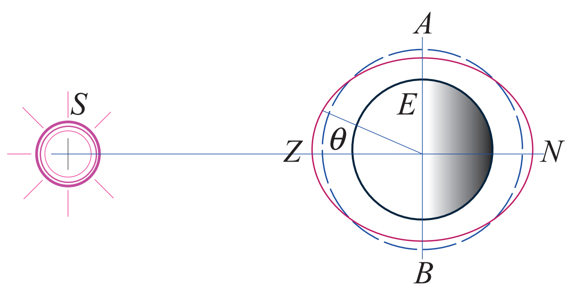

Figure 4: The equilibrium static distortion of the water surface

Hence we can write for the small inclination α of the water surface with respect to the horizon:

$$\alpha = \frac{1}{r}\frac{dr(\theta)}{d\theta} \approx -\frac{2a}{r_0}\sin 2\theta. \tag{10}$$

We see that the water surface is horizontal ($\alpha = 0$) at locations $\theta = 0$ and $\theta = 90°$ (points Z and A, see Figure 4). The angle α is a maximum at points where the tidal force is directed horizontally, namely at $\theta = \pm 45°$ and at $\theta = \pm 135°$, and equals $2a/r_0$. In equilibrium the distorted water surface is orthogonal to the plumb line. The plumb line (on a non-rotating planet) shows the direction of the vector sum of the planet's gravity and the tidal force. A small departure of the plumb line from the direction of the planet's gravity is caused by the horizontal component of the tidal force. Therefore, the angle α equals the ratio of the horizontal tidal force $(F_{\text{tid}})_{\text{hor}}$ to the force of the planet's gravity $F_{\text{grav}} = mg$. If we equate $\alpha = 2a/r_0$ at $\theta = 45°$ to $(F_{\text{tid}})_{\text{hor}}/F_{\text{grav}}$ and take into account that for sun-induced tides, $(F_{\text{tid}})_{\text{hor}}/F_{\text{grav}} = (3/2)(M_{\text{sun}}/M_{\text{earth}})(r_0^3/R^3)$, we find for the maximum static level difference $2a$ at points Z and A the following value:

$$2a = \frac{3}{2}r_0\frac{M_{\text{sun}}}{M_{\text{earth}}}\frac{r_0^3}{R^3}. \tag{11}$$

Equation (11) yields $2a = 0.24\,\mathrm{m}$. A similar expression is valid for the static distortion of the ocean surface due to the lunar tidal force, and yields $2a = 0.54\,\mathrm{m}$ for the moon-induced static distortion.

2.4 Tidal forces on the rotating earth

In the above treatment of tide-generating forces we have used a revolving but non-rotating geocentric reference frame. The origin of this frame moves in a circle around the sun-earth (moon-earth) center of mass, but the frame itself does not rotate because the directions of its axes are fixed relative to the distant stars. That is, the frame moves translationally in a circle. This reference frame is convenient for the analysis of motion of an earth's artificial satellite. If we ignore the perturbations caused by tidal forces, the earth's satellite traces out a closed elliptical orbit relative to this reference frame.

To introduce tidal forces on the rotating earth, we must use a true geocentric frame of reference that takes part in the daily rotation (spin) of the earth. This frame is non-inertial, and hence we should be concerned with the acceleration of its different points. We can consider the motion of the earth (and of the true geocentric reference frame) as consisting of two components. The first is the component considered above, namely the translational motion (revolution without rotation) about the sun-earth (moon-earth) center of mass under the gravitational forces. The second component is a uniform daily rotation (spin) of the earth about an axis passing through the center of the earth.

Both these motions of the earth are important in the problem of tides, but the roles they play are quite different. The acceleration $\mathbf{a}_0$ related to the translational motion of the earth under the external gravitational forces is responsible for the origin of the uniform pseudo force of inertia $\mathbf{F}_{\mathrm{in}} = -m\mathbf{a}_0$, whose action on a body on the earth, combined with the non-uniform gravitational pull of the sun (moon), is described by the above considered tidal force $\mathbf{F}_{\mathrm{tid}}$. We note again that only the acceleration $\mathbf{a}_0 = GM_{\mathrm{sun}}/R^2$ of this translational motion is important, not the orbital velocity of the earth.

To avoid confusion often encountered in the literature, we must be careful with the terminology and definitions. A tide is a distortion in the shape of one body induced by the gravitation of another celestial object. In discussing tides, we should be concerned only with those gravitational and inertial forces that depend on the apparent position of the celestial body that produces the tide.

Due to the axial rotation of the earth, its points are moving with centripetal accelerations directed towards the axis. These centripetal accelerations give rise to centrifugal pseudo forces that increase in proportion to the distance from the earth's axis. These forces have nothing to do with tides. In order to avoid confusion, we shouldn't mix these centrifugal pseudo forces of daily rotation with the considered above translational pseudo force of inertia.

The centrifugal force produced by the earth's daily rotation is much greater in magnitude than tidal forces. Because of the centrifugal forces, the equilibrium shape of the earth (with the oceans that cover it) differs from an ideal sphere—it is approximately an ellipsoid of rotation (spheroid) whose equatorial diameter is a bit greater (about 45 km) than the polar diameter. The centrifugal effect of the earth's daily rotation causes an equatorial bulge, which is the principal departure of the earth's shape from an ideal sphere. But we must distinguish such earth shape distortions from tidal effects.

Discussing the tides, we are not concerned with this constant distortion of the earth, because this distortion is independent of the apparent position of the celestial body that produces the tides. The equatorial bulge isn't due to gravitational fields of an external mass, and it has no periodic variations. Actually, this oblate shape is the reference baseline against which real tidal effects are measured. Analyzing the origin of tides, we have no need to take this distortion into account. We emphasize again that the centripetal acceleration of the axial rotation adds nothing to tidal forces.

However, the daily rotation of the earth makes tidal forces time-dependent because the pattern of tidal forces on the earth is coupled to the apparent positions of the sun and moon. A dynamical response of the oceanic waters to these time-dependent forces on the spinning earth is the essence of the phenomenon of tides. For an observer on the earth, the system of tide-generating forces, coupled to the apparent position of the source, rotates as a whole (rotates "rigidly") with angular velocity Ω of the earth's daily rotation, following the changing apparent position of the celestial body in question.

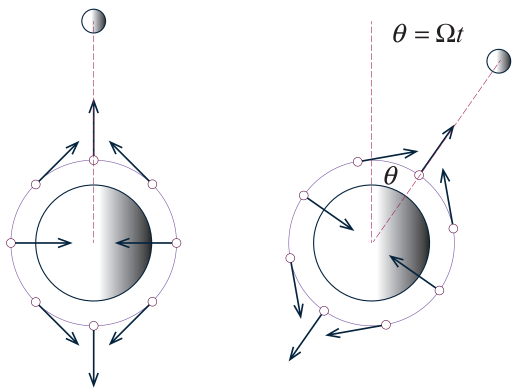

Figure 5: The quadrupole configuration of tide-generating forces, rotating as a whole in accordance with the changing apparent position of the moon.

This daily rotation of the tide-generating forces is illustrated by Figure 5, and much more clearly by the simulation program "The Oceanic Tides" developed by the author (Butikov, 2012). Although the sun-induced system of tidal forces rotates as a whole with the angular velocity Ω of the earth's axial rotation, that is, with a period of $2\pi/\Omega$, the true period T of variation of the tidal forces on the earth equals half this value ($T = \pi/\Omega$) because of the quadrupole symmetry of the system of forces (the semidiurnal tide). For the sun-induced tidal forces the period equals 12 hours. For the moon-induced tidal forces the period is a bit longer (12 hours 25 minutes). The difference between the two periods is due to the orbital motion of the moon, which is also influential on its apparent position.

To keep the things simple, further on we will consider the case in which the celestial body that causes the tides (sun or moon) is in the equatorial plane of the earth. If we fix a point on the equator of the earth, the local tidal force vector executes a uniform rotation in the vertical plane, making two complete revolutions during a day. The simulation (Butikov, 2012) clearly shows how the daily rotation of the whole system of tidal forces produces this doubly-fast uniform rotation of the tidal force at a given equatorial point, as seen by an observer on the spinning earth.

The changes in directions of the tidal forces at different fixed points are shown in Figure 6 (b – for only 4 fixed points, c – for 8 fixed points). When the apparent direction towards the sun (or moon) changes through an angle θ, each of the tidal force vectors

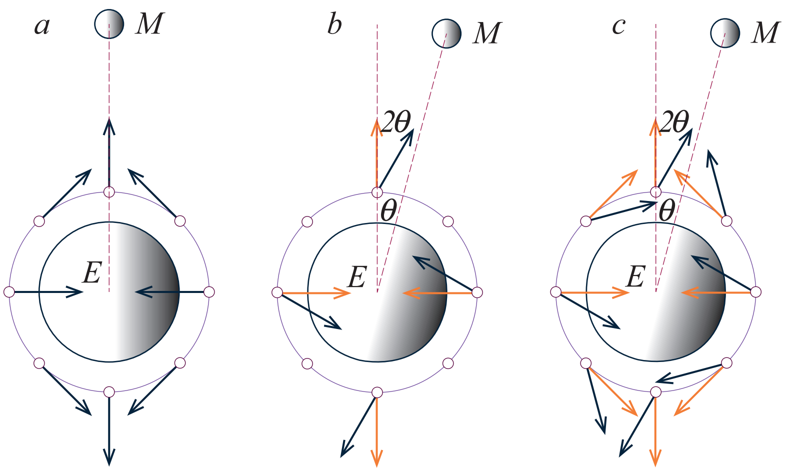

Figure 6: The changes in directions of the tidal forces at different fixed points in accordance with the changing apparent position of the moon.

at fixed points turns in the same direction through the twice as great angle 2θ. We note that for any fixed point on the earth the tidal force vector in Figure 6, b–c makes the same angle 2θ with the direction of tidal force at this particular point in Figure 6, a.

At the opposite points of the equator, the tidal force vectors at any time moment are directed oppositely, that is, they are rotating in the opposite phases. At any point of the equator the tidal force is directed upward twice during a day. This occurs when the sun (or moon) for this point is at the zenith and nadir. Also twice a day the tidal force is directed downward. This occurs when the sun (or moon) for this point is at the horizon.

Because of this periodic dependence on the time, the tidal forces, in spite of their small magnitude compared even to the centrifugal force of inertia caused by the earth's daily rotation, produce magnificent oceanic tides.

2.5 Decomposition of rotating tidal forces onto oscillating forces

To obtain the dynamical picture of the oceanic tides on the rotating earth, we should use the reference frame that rotates with the earth. As we have seen above, relative to this frame the quadrupole system of tide-generating forces, being coupled to the position of the sun (moon), rotates as a whole while the sun (moon) travels along its apparent daily path around the earth. This rotation of the whole system of forces occurs at an angular velocity Ω, the angular velocity of the earth's daily rotation (or the difference between Ω and the angular velocity of the moon in its orbit for moon-induced tides).

In the problem of tides, expressions for the components of tide-generating forces $(F_{\rm tid})_{\rm hor}$ and $(F_{\rm tid})_{\rm vert}$ in Eqs. (7) are applicable also to the true geocentric frame of reference, which takes part in the daily axial rotation of the earth. To find analytical expressions for the time dependence of the tidal forces at a given point in the equatorial plane of the spinning earth, we substitute $\theta(t) = \Omega t$ in Eqs. (7) for $(F_{\rm tid})_{\rm hor}$ and $(F_{\rm tid})_{\rm vert}$. This substitution yields the following expressions for the point of the equator at which the sun culminates (passes through its zenith) at $t = 0$:

$$(F_{\rm tid})_{\rm vert}(t) = Ar\cos(2\Omega t); \quad (F_{\rm tid})_{\rm hor}(t) = -Ar\sin(2\Omega t), \tag{12}$$

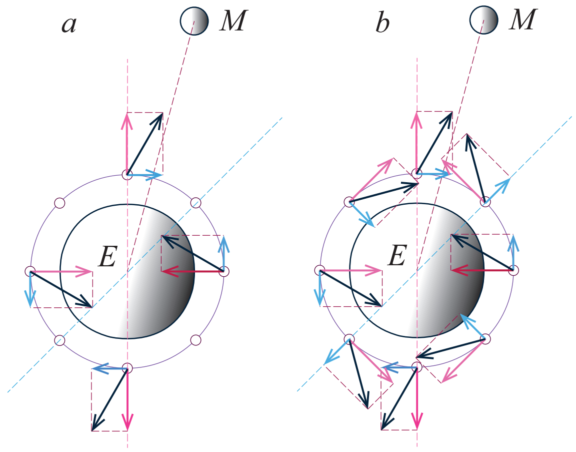

Figure 7: Decomposition of rotating tidal forces onto oscillating vertical and horizontal components for 4 fixed equatorial points (a) and for 8 points (b).

where $A = (3/2)F_{\rm gr}/R = (3/2)GmM_{\rm sun}/R^3$. At any other equatorial point of the earth, the tidal force vector also rotates in the vertical plane with angular velocity 2Ω. That is, all the vectors of tidal forces at different points rotate synchronously but with different phases (Figure 6, a–c).

Equations (12) show that at point Z (and at any other point of the equator) the rotating vector of the tidal force can be considered as a superposition of two forces of fixed directions (vertical and horizontal), which oscillate with frequency 2Ω and a phase shift of quarter period. This decomposition of rotating tidal forces is illustrated by Figure 7, a–b for 4 fixed equatorial points and for 8 points, respectively.

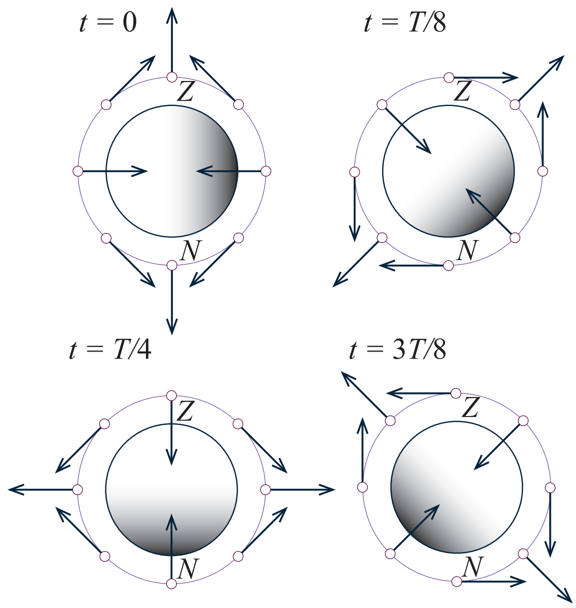

Figure 8: Two quadrupole systems of oscillating forces (left and right columns) that do not rotate. Their axes of symmetry make an angle $\pi/4$ to one another.

Since this decomposition of a uniformly rotating vector onto two orthogonal oscillating vectors is valid for any point of the equator, the whole rotating system of tidal forces can be represented as a superposition of two quadrupole systems of forces oscillating with a phase shift of quarter period. These two oscillating systems (left and right columns of Figure 8, respectively) do not rotate; their axes of symmetry make an angle $\pi/4$ to one another. At each point the force of one of these systems oscillates along the radial (vertical) direction, while the force of the other system—along the tangential (horizontal) direction. The oscillations of these orthogonal components occur a quarter period out of phase. At $t = 0$ and at $t = T/4$ the forces of one system (left in Figure 8) reach their maximum, while the forces of the other system (right) turn to zero, and vice versa at $t = T/8$ and $t = 3T/8$. At any given point in the equatorial plane, the vector sum of these mutually orthogonal oscillating forces produces a force of constant magnitude whose direction rotates uniformly with angular velocity $\omega = 2\Omega$. The time sequence $0 \to T/8 \to T/4 \to 3T/8$ in Figure 8 shows how during $T/2$ each force executes one full revolution. For different points on the earth, the phases of these rotating vectors differ.

This behavior of the tidal forces can be clearly observed with the help of the above mentioned simulation program (Butikov, 2012) developed by the author and available on the web. Such a representation of rotating tidal forces as the two systems of oscillating forces with the axes of symmetry at an angle $\pi/4$ to one another will help us in understanding the tidal wave, which circulates around the globe, as a result of superposition of two steady-state forced oscillations of the ocean water.

We note again that the first part of the problem of tides, namely the origin and properties of tide-generating forces, can be completely solved rigorously within the framework of Newtonian mechanics, as we have shown above.

3 The tides as steady-state forced oscillations

Most textbook authors oversimplify the problem of tides and consider (after Newton and Bernoulli) only the so-called static (or equilibrium) theory of tides, which treats the ocean surface as a liquid ellipsoid stretched along the earth-moon line (or earth-sun line for sun-induced tides), as if this surface were always in equilibrium under the earth's force of gravity and tidal forces produced by the moon (or sun). In this approach, the tidal bulges are aligned with the earth-moon (or earth-sun) axis. Therefore on the spinning earth the moments of high water at a given location should coincide with the upper and lower culminations of the moon (sun), that is, when the moon (sun) passes through its zenith and nadir. Observations do not agree with this prediction. Instead, almost the opposite is usually observed, at least in the open ocean: the moments of low tide occur approximately at the culminations of the moon.

A complete theory of the tides should take into account the dynamical response of the ocean to the time-dependent tide-generating forces. The dynamical theory of tides (first suggested by Laplace and later developed by Airy) treats the tides as a steady-state forced motion (under varying tidal forces) of a dynamical system (the ocean). Such a theory predicts a resonant growth of the steady-state amplitude in cases when the driving period approaches the period of natural oscillations.

3.1 The natural standing wave

To avoid the complications related to the three-dimensional character of the problem and to explain the physical aspect of the dynamical theory using the simplest possible model, we imagine, following Airy, water in a wide canal of uniform depth engirdling the entire earth along the equator.

Imagine the water surface in this canal being distorted statically by some quadrupole system of forces (like the tide-generating forces), so that two bulges are formed on the opposite sides of the earth, changing the shape of the water surface from circular to elliptical. This distorted shape of the water surface is shown (strongly exaggerated) on the left panel of Figure 9.

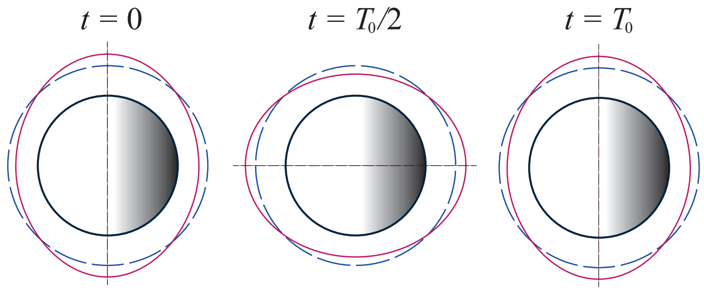

Figure 9: Quadrupole oscillations of the water level in a wide canal of uniform depth engirdling the entire earth along the equator.

If the forces maintaining this shape suddenly vanish, the earth's gravity would make the distorted surface restore its equilibrium, circular shape. The water would start to flow and the bulges disappear so that after a time, namely a quarter period, the water surface would become circular. But because the water continues to move, after another quarter period the bulges reappear in new positions showing an elliptical distortion of the surface along the line perpendicular to the line of the original distortion (central panel of Figure 9). Then the motion repeats itself in reverse (right panel of Figure 9). This motion of water in the circular canal is a standing gravitational surface wave whose wavelength equals half-circumference of the globe. Such a mode of oscillation is characterized by a certain natural period T_0.

The superposition of two such standing waves whose phases differ by $\pi/2$ and whose elliptical axes are separated by 45° produces a circulating (traveling) wave of constant elliptical shape and a wavelength equal to half of the earth's circumference. The two opposite bulges in the water surface travel with this wave around the globe preserving their height and shape.

An essential point in explaining the steady-state phase shift between the moments of high tide and the culmination of the moon (sun) is the relation between the natural period T_0 of this circulating wave and the period T of the tide-generating driving forces. It is possible to estimate T_0 as the time taken by the circulating surface wave to travel along half the globe. In the limiting case of very long waves on the surface of shallow water ($\lambda \gg h$) the speed of wave is determined by the earth's gravity g and depth h, and is independent of the wavelength λ. From hydrodynamics we know that this speed equals $\sqrt{gh}$. We assume that the mean value h of the ocean depth is 3.5 km. During a period T_0

the wave travels half the circumference of the globe πr_0, and hence $T_0 = \pi r_0/\sqrt{gh} \approx$ 30 hours. Thus, the approximately 12-hour driving external period T is less than natural period T_0 of the water free oscillation in the equatorial canal surrounding the earth.

We emphasize that the water of the ocean takes part in the daily rotation of the earth. It is the shape of the water surface (the wave) that is moving with respect to the rotating earth. Relative to the earth, water particles on the surface of the ocean execute oscillatory motions in closed paths that are considerably stretched horizontally. These back and forth motions constitute the tidal currents. On the average, the water is stationary in the rotating geocentric frame.

3.2 The tides as steady-state forced oscillations of the ocean

Turning to the second part of the discussed physical problem, we can consider the tides in the equatorial canal as steady-state forced oscillation of the water surface due to the well-known time-dependent tidal forces. Each of the two above-described oscillating systems of forces excites a mode of forced oscillation of the water, specifically the mode of the same symmetry as is characteristic of the corresponding system of driving forces. These modes have elliptical shapes, much like the above considered natural oscillations, namely, the elliptical standing waves whose axes make an angle of 45° with one another. We can consider these modes to be orthogonal in the sense that their spatial forms are described by eigenfunctions forming an orthogonal two-dimensional basis in the function space. Each mode is characterized by a single normal coordinate whose time dependence is found as the periodic solution to the differential equation describing the forced motion of an ordinary linear damped oscillator.

Any steady-state forced oscillation occurs exactly with the period of the driving force. The amplitude and phase lag of the oscillation depend on the amplitude of the driving force, on the damping factor, and, more importantly, on the relation between the driving and natural periods. The two systems of oscillating driving tidal forces are characterized by equal amplitudes and frequencies. Also the natural frequencies and damping factors of both excited modes are equal. Hence both excited modes also have equal amplitudes and equal phase delays behind the corresponding driving forces.

If we ignore friction (dissipation of mechanical energy in the excited oscillation), the forced motion occurs exactly in phase with the driving force, provided the driving period T is longer than the natural period ($T > T_0$). Otherwise (if $T < T_0$) the forced motion occurs in the opposite phase with respect to the driving force. Such oscillations (at $T < T_0$) of the water level in a wide canal are shown by left and right columns in Figure 10. Each column corresponds to the relevant system of oscillating forces shown in left and right columns of Figure 8, respectively. With friction, there is a phase lag of forced water oscillations behind the driving forces. The amplitude of these forced oscillations and the phase lag are discussed in Appendix II.

The two forced oscillations in this linear system, each excited by one system of driving tidal forces, are independent of one another, and the resulting forced motion is a superposition of these forced oscillations. The superposition of these two oscillatory modes produces a forced circulating (traveling) elliptical wave. The time sequence $0 \to T/8 \to T/4 \to 3T/8$ in Figure 10 shows how during $T/2$ the resulting elliptical wave makes a half-revolution (clockwise) around the globe. This behavior of the traveling tidal bulges is very clearly illustrated by the simulation program (Butikov, 2012).

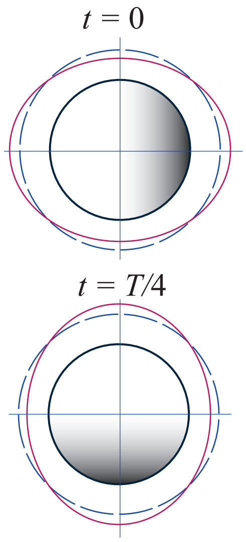

Figure 10: Quadrupole steady-state oscillations of the water level in a wide canal forced by the two systems of oscillating forces shown in left and right columns of Figure 8.

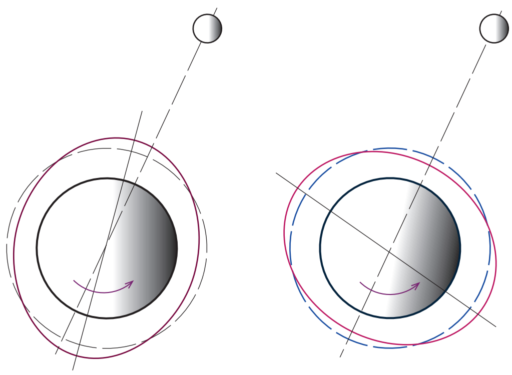

Figure 11: The celestial body that is the cause of tides and the circulating tidal wave at $T_0 < T$ (left panel), and at $T_0 > T$ (right panel).

Distortion of the water surface in the circulating tidal wave forced by a celestial body is illustrated by Figure 11. The distortion is strongly exaggerated. Due to the earth's axial rotation (counter-clockwise), the apparent position of the celestial body moves clockwise. The water bulges are almost aligned with the direction to apparent position of the celestial body at $T_0 < T$ (left panel); a small lag is caused by friction. At $T_0 > T$ (right panel) the two bulges are aligned almost perpendicularly to this direction.

For the simplified model of tides in the equatorial canal of uniform depth (and also for an earth covered everywhere by an ocean of uniform depth), the natural period of free oscillation is longer than the 12-hour driving period. Thus the dynamical theory predicts in this case a stationary circulating elliptically shaped wave whose axis (the line of tidal bulges) is almost perpendicular to the earth-sun (earth-moon) line.

On the other hand, the natural period of an elastic wave in the crust of the earth is shorter than the 12-hour period of the tidal forces. Hence, in the frictionless model, bulges in the earth's crust are oriented along the earth-sun (earth-moon) line. Observations show that the solid body of the earth actually experiences twice-daily tides with maximum amplitude of about 30 cm whose bulges lag approximately 3° behind the earth-moon line. For more details on the circulating tidal wave see Appendix II.

In the above discussion, we considered only the steady-state oscillation of the ocean surface (a stationary wave), assuming that the transient is already over. For this steady motion to establish itself, some friction (even if very small) is necessary. In the problem of tides, we are concerned with the water motion caused solely by the eternally lasting tidal forces, and therefore we have had centuries and even millennia for any transient to fade away. Therefore our use of the steady-state solution is appropriate for tides. Again we emphasize that in the dynamical theory of tides, the driving tide-generating forces are perfectly well known, so that most uncertainties originate from a very poor correspondence between the accepted simple model of the dynamical system and the real oceans of the earth.

3.3 Beyond the simple model: Real-world complications

The pattern of lunar tide-generating forces is coupled to the apparent position of the moon with respect to the earth. The relative position of the moon has an average periodicity of 24 hours 50 minutes. The lunar tide-generating force experienced at any location has the same periodicity. When the moon is in the plane of the equator, the force runs through two identical cycles within this time interval because of the quadrupole symmetry of the global pattern of tidal forces. Consequently, the tidal period is 12 hours 25 minutes in this case (the semidiurnal lunar tide). However, the lunar orbit doesn't lie in the plane of the equator, and the moon is alternately to the north and to the south of the equator. The daily rotation of the earth about an axis inclined to the lunar orbital plane introduces an asymmetry in the tides. This asymmetry reveals itself as an inequality of the two successive cycles within 24 hours 50 minutes.

Similarly, the sun causes a semidiurnal solar tide with a 12-hour period, and a diurnal solar tide with a 24-hour period. The interference of the sun-induced tidal forces with the moon-induced tidal forces (the lunar forces are about 2.2 times as strong) causes the regular variation of the tidal range between the spring tide, when the range has its maximum (occurring at a new moon and at a full moon, when the sun and moon are in the same or in the opposite directions and their tide-generating forces are aligned), and neap tide, when the range has its minimum (which occurs at intermediate phases of the moon). The amplitude of a spring tide may be 2.7 times the amplitude of a neap tide.

In a complete description of the local variations of the tidal forces, still other partial tides play a role because of further inequalities in the orbital motions of the moon and the earth. In particular, the elliptical shape of the moon's orbit produces a 40 percent difference between the lunar tidal forces at the perigee and apogee of the orbit. Also the inclination of the moon's orbit varies periodically in the interval 18.3° – 28.6°, causing a partial tide with a period of 18.6 years.

The most serious complications in the observed picture of tides are related to complexity of the dynamical system that responds to fairly well known tidal forces. The earth is not surrounded by an uninterrupted water envelope of equal depth, but rather has a very irregular geographic alternation of land and seas with complex floor geometry. The actual response of the oceans and seas to the tidal forces is extremely complex. In enclosures formed by gulfs and bays, the local tide is generated by an interaction with the tides of the adjacent open ocean. Such a tide often takes the form of a running tidal wave that circulates within the confines of the enclosure. In some nearly enclosed seas, such as the Mediterranean and Baltic seas, the tidal forces may generate an oscillation in the form of a standing wave, or tidal seiche. In these seas, the tidal range of sea level is only on the order of centimeters. In the open ocean, it generally is on the order of decimeters.

In bays and adjacent seas, however, the tidal range may be much greater because the shape of a bay may favor the enhancement of the tide inside. In particular, there may be a resonance response of the basin to the tidal forces.

Tides are most easily observed along seacoasts, where the amplitudes of water level are exaggerated. When tidal currents run into the shallow waters of the continental shelf, their rate of advance is reduced, the energy accumulates in a smaller volume, and the rise and fall are amplified.

Peculiarities of tidal motions in coastal waters, particularly in gulfs and estuaries, depend on the details of coastal geometry and water-depth variation over a complex sea floor. Tidal amplitudes and phase lags, the contrast between spring and neap tides, and the variation of times of high and low tide all change widely from place to place.

For the aforementioned reasons, a purely theoretical calculation of the times and heights of tides at a particular location is practically impossible. Nevertheless, for a given place on a coast, the tides can be quite successfully predicted on the basis of accumulated long-term observations of the tides at this place. The analysis of the observations relies on the fact that any tidal pattern is a superposition of variations associated with known periodicities in the motions of the moon and sun relative to the earth.

As we have seen above, in the absence of friction the phase shift between the variation of the driving forces and the steady-state forced oscillations in the open ocean is zero below resonance and equals half-period over resonance. In the first case the tidal bulges are oriented along the earth-sun (or earth-moon) line, while in the second -perpendicularly to this line. In both cases the net torque of the gravitational force exerted on the earth equals zero.

The tidal friction destroys the symmetric configuration of the tidal bulges, making them deviate from the line earth-moon or from the perpendicular line. Due to this deviation a retarding torque appears, giving rise to a gradual slowing down of the earth's axial rotation. Simultaneously the moon's orbit gradually expands. Such a heavenly evolution eventually synchronizes the axial rotation with the orbital motion of both celestial bodies coupled by mutual gravitation. Theoretical background and further details of the phenomena caused by tidal friction can be found, for example, in our previous paper (Butikov, 2002).

4 Concluding discussion

Tidal forces exerted on the earth and the earthly bodies result from gravitation of other celestial bodies, primarily of the moon and sun. In the origin of tidal forces, the gravitational field itself doesn't play the decisive role, but rather the non-uniformity of this field throughout the globe (the differential gravitation). Since we are interested in tidal forces on the earth, the geocentric reference frame is the most suitable for their description and for understanding the underlying physics. This is a non-inertial reference frame, so that in the framework of Newtonian mechanics any body in this frame is subjected to pseudo forces of inertia, in addition to all other real forces of interaction with the surrounding bodies.

The tidal force exerted on any body on the earth equals the vector sum of its gravitational attraction by the celestial body (moon or sun) and the uniform force of inertia associated with *translational* acceleration of the earth due to its attraction as a whole by the celestial body in question. This force of inertia is independent of the orbital velocity of the earth in its motion around the sun (or in its whirling together with the moon about their common center of mass). For different positions near the earth, the tidal forces form a quadrupole system with the axis of symmetry along the line joining the earth center with the apparent position of the celestial body. If the earth were not spinning, its water shell would have under these tidal forces (and under the earth's gravity) an equilibrium spheroidal shape stretched slightly along this line.

For an observer on the spinning earth, this quadrupole system of tidal forces rotates as a whole with the angular velocity of the earth's axial rotation. At any fixed equatorial point on the earth, the tidal force (produced by the moon or sun when it occurs over the equator) rotates in the vertical plane with twice as great angular velocity. The system of these rotating vectors can be represented as a superposition of two non-rotating quadrupole systems of forces with axes of symmetry making an angle of 45° with one another, and oscillating a quarter period out of phase. Each of these systems causes a steady-state forced oscillation of the oceanic water. In a simplified model of the ocean as a water shell of equal depth this oscillation is a standing wave of elliptical shape whose amplitude and phase are determined by the well-known steady-state solution to the equation of a linear oscillator. The tidal wave circulating around the globe is a superposition of these two standing waves.

The real-world picture of tides at a certain location is much more complex not only due to interference of the sun-induced and moon-induced tidal forces, but, more importantly, due to the continents with complex details of coastal geometry and depth variations over a complex sea floor. The real ocean, compared to the simplified model adopted above, is a very complex dynamical system. This makes a purely theoretical calculation of its response to the well-known tidal forces extremely difficult.

Appendix I. Tidal forces at an arbitrary point near the earth

To emphasize the physics underlying the origin of tide-generating forces, we consider the vectorial addition of the relevant forces instead of the standard derivation that uses the tide-generating potential, for which the mathematics is somewhat simpler.

To obtain a general mathematical expression for the sun-induced tidal force at an arbitrary point H over the earth (see Figure 3), we introduce the radius vector $\mathbf{r}$ of this point relative to the center of the earth, and its radius vector $\mathbf{r}_s = \mathbf{R} + \mathbf{r}$ relative to the center of the sun S, where $\mathbf{R}$ is the radius vector of the earth relative to the sun. The tidal force $\mathbf{F}_{\rm tid}$ experienced by a body of mass m at point H (in the non-inertial geocentric frame) is the resultant of its gravitational attraction $\mathbf{F}_{\rm gr} = -GmM_{\rm sun}\mathbf{r}_s/r_s^3$ to the sun and the pseudo force of inertia $\mathbf{F}_{\rm in} = -m\mathbf{a}_0 = GmM_{\rm sun}\mathbf{R}/R^3$:

$$\mathbf{F}_{\rm tid} = \mathbf{F}_{\rm gr} + \mathbf{F}_{\rm in} = -GmM_{\rm sun}\left(\frac{\mathbf{r}_s}{r_s^3} - \frac{\mathbf{R}}{R^3}\right). \quad (13)$$

Next we express $\mathbf{r}_s$ in Eq. (13) as the sum $\mathbf{R}+\mathbf{r}$ and calculate the square of $\mathbf{r}_s$ taking into account that $r \ll R$:

$$r_s^2 = (\mathbf{R}+\mathbf{r})^2 = R^2 + 2(\mathbf{R}\cdot\mathbf{r}) + r^2 \approx R^2\left(1 + 2\frac{(\mathbf{R}\cdot\mathbf{r})}{R^2}\right).$$

Therefore,

$$\frac{1}{r_s^3} \approx \frac{1}{R^3}\left(1 - 3\frac{(\mathbf{R}\cdot\mathbf{r})}{R^2}\right).$$

Substituting this expression for $1/r_s^3$ into Eq. (13) yields:

$$\mathbf{F}_{\rm tid} \approx -G\frac{mM_{\rm sun}}{R^3}\left[(\mathbf{R}+\mathbf{r})\left(1 - 3\frac{(\mathbf{R}\cdot\mathbf{r})}{R^2}\right) - \mathbf{R}\right]$$
$$\approx -G\frac{mM_{\rm sun}}{R^3}\left[\mathbf{r} - 3\mathbf{R}\frac{(\mathbf{R}\cdot\mathbf{r})}{R^2}\right]. \quad (14)$$

We note that the main contributions of $\mathbf{F}_{\rm gr}$ and $\mathbf{F}_{\rm in}$ to $\mathbf{F}_{\rm tid}$, whose magnitudes are inversely proportional to R^2, cancel in Eq. (14). This cancelation corresponds to the aforementioned state of weightlessness that we experience on the spaceship 'Earth' with respect to the sun's gravity. The magnitude of the remaining in Eq. (14) force is inversely proportional to R^3, that is, to *cube* of the distance from the sun (generally, from the celestial body which is the source of the tidal force).

For the points A and B considered in elementary treatment of the tidal forces in Section 2.1 (see Figure 1), radius vector $\mathbf{r}$ is perpendicular to $\mathbf{R}$ and the scalar product $\mathbf{R}\cdot\mathbf{r}$ in Eq. (14) is zero. Hence at these points the tidal force is directed opposite to $\mathbf{r}$ (that is, vertically downward) and its magnitude equals $GmM_{\rm sun}r/R^3$. For the points Z and N the tidal force, according to Eq. (14), is directed along $\mathbf{r}$ (that is, vertically upward), and its magnitude $2GmM_{\rm sun}r/R^3$ is two times greater than at points A and B. We see that at these four points, the general result given by Eq. (14) agrees with the above presented simpler calculations, see Eqs. (1) and (2).

Appendix II. The amplitude and phase lag of the tidal wave

Next we present a mathematical description of forced steady-state oscillations of water in the equatorial canal under the oscillating tidal forces. Each of the partial forced oscillations of the water surface in the canal (each normal mode) can be described by a differential equation of a linear oscillator. Let $q_1(t)$ be the normal coordinate describing the first forced oscillation whose elliptical shape is characterized by a major axis oriented along $\theta = 0$, that is, along the earth-sun (earth-moon) line (and in the perpendicular direction after a half period), and let $q_2(t)$ be the normal coordinate describing the second oscillation with the axis inclined through 45° with respect to the earth-sun line. A disturbance of the water surface caused by the first oscillation can be described by $\Delta r_1(\theta, t) = q_1(t)\cos(2\theta)$, which gives the small vertical displacement of the surface at an arbitrary point (r_0, θ) of the equator. Similarly, the second oscillation causes a distortion of the surface described by $\Delta r_2(\theta, t) = q_2(t)\sin(2\theta)$. The forced oscillations experienced by the normal coordinates $q_1(t)$ and $q_2(t)$ are periodic (steady-state) partial solutions of the two differential equations:

$$\begin{aligned} \ddot{q}_1 + 2\gamma\dot{q}_1 + \omega_0^2 q_1 &= \omega_0^2 a\cos\omega t, \\ \ddot{q}_2 + 2\gamma\dot{q}_2 + \omega_0^2 q_2 &= \omega_0^2 a\sin\omega t. \end{aligned} \tag{15}$$

Here ω_0 is the natural frequency of the corresponding mode ($\omega_0 = 2\pi/T_0 = 2\sqrt{gh}/r_0$), γ is the damping constant, $\omega = 2\Omega$ (twice the angular velocity Ω of the earth's daily rotation) is the driving frequency, and a is the magnitude of the equilibrium distortion of the ocean surface under the static system of tidal forces (that is, the distortion for the hypothetical planet whose axial rotation is synchronized with its orbital revolution). The theoretical value of a is given by Eq. (11), Section 2.3. Although the values of a and of $\omega = 2\Omega$ that characterize the driving tidal forces are fairly well known, the situation is quite different regarding the values of ω_0 and γ, which characterize the ocean as a dynamical system.

In the hypothetical limiting case of extremely slow rotation of the earth, the steady-state solution of Eqs. (15) is $q_1(t) = a\cos\omega t = a\cos 2\Omega t,\ q_2(t) = a\sin\omega t = a\sin 2\Omega t$. This solution describes the quasi-static elliptical distortion whose axis follows adiabatically the slowly rotating earth-sun (earth-moon) line. The major axis of the ellipse at any moment is oriented along this line. The displacement of the water level from its mean position in the equatorial plane in this limiting case is given by:

$$\begin{aligned} \Delta r(\theta, t) = \Delta r_1(\theta, t) + \Delta r_2(\theta, t) = q_1(t)\cos 2\theta + q_2(t)\sin 2\theta = \\ a(\cos 2\Omega t\cos 2\theta + \sin 2\Omega t\sin 2\theta) = a\cos 2(\Omega t - \theta). \end{aligned} \tag{16}$$

To find the distortion of the water surface for an arbitrary value of ω, we can use the well-known relevant steady-state solutions to Eqs. (15) for the normal coordinates $q_1(t)$ and $q_2(t)$:

$$q_1(t) = q_0\cos(\omega t - \delta), \quad q_2(t) = q_0\sin(\omega t - \delta), \tag{17}$$

where their common amplitude q_0 and phase lag δ are given by the following expressions:

$$q_0 = \frac{\omega_0^2 a}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}, \qquad \tan\delta = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}. \tag{18}$$

Therefore the resulting distortion of the water surface under the tidal forces is given by

$$\begin{aligned} \Delta r(\theta, t) = \Delta r_1(\theta, t) + \Delta r_2(\theta, t) = q_1(t)\cos 2\theta + q_2(t)\sin 2\theta = \\ q_0[\cos(2\Omega t - \delta)\cos 2\theta + \sin(2\Omega t - \delta)\sin 2\theta] = \\ q_0\cos 2(\Omega t - \delta/2 - \theta). \end{aligned} \tag{19}$$

We see from Eq. (19) that at any time t the maximum of the tidal wave circulating around the earth (high water) is located at the position defined by the angle $\theta_{\max} = \Omega t - \delta/2$. That is, the position of this maximum lags behind the sun (moon) by the angle $\delta/2$. If $\gamma \ll \omega$, it follows from Eq. (19) that this retarding angle is almost zero for the case $\omega < \omega_0$. In other words, the marine tide would be nearly the equilibrium tide with the high-water time coinciding with culminations of the sun (moon) if the natural period of the circulating wave were less than the 12-hour driving period (that is, if $T_0 < T$). However, for our model of the ocean, we estimate the natural period to be approximately 30 hours. Therefore the situation corresponds to $\omega > \omega_0$ ($T < T_0$), when the steady-state forced oscillations occur nearly in the opposite phase relative to the driving force. In this case the tide should be inverted with respect to the equilibrium one. The retarding angle $\delta/2$ approaches $\pi/2$ according to Eq. (19), which means that for a given equatorial point, the high water occurs when the sun (moon) is almost at the horizon (rather than at zenith or nadir).

At any given place on the equator, it follows from Eq. (19) that the water level (above the average value) varies with time t according to $z(t) = q_0 \cos(2\Omega t - \delta)$, where $t = 0$ corresponds to the culmination of the sun (moon) at the place in question. We can expect that for the model of a water canal of uniform depth, the value of q_0 given by Eq. (18) is more or less reliable because hydrodynamics allows us to estimate the natural frequency $\omega_0 = 2\pi/T_0 = 2\sqrt{gh}/r_0$ by using the known speed $v = \sqrt{gh}$ of very long gravitational waves. The situation is far from resonance. This means that the amplitude of the tidal wave in the canal (and in the open ocean) has the same order of magnitude as the static (equilibrium) distortion given by the above estimates (Eq. (11), Section 2.3).

However, considerable uncertainty is related to the damping factor γ. If we assume that the damping is small ($\gamma \ll \omega_0$), we can conclude that the orientation of the tidal bulges deviates only slightly from the line perpendicular to the moon-earth line, but the particular value of this deviation remains indefinite.

References

[1] Eugene I. Butikov (2002), "A Dynamical Picture of the Oceanic Tides." *Am. J. Phys.* v. **70** pp. 1001 – 1011.

[2] Eugene Butikov (2012), "The Oceanic Tides." A simulation program (Java applets), available on the web at http://faculty.ifmo.ru/butikov/Projects/Tides0.html

[3] NOAA (2016), National Ocean Service. "Our Restless Tides." https://www.co-ops.nos.noaa.gov/restles3.html. Accessed 21.12.2016.

[4] D. A. Ross (1995), *Introduction to Oceanography*. New York, NY: HarperCollins pp. 236–242.

[5] Mikolaj Sawicki (1999), "Myths about Gravity and Tides." *The Physics Teacher*, **37**, pp. 438–441. A pdf revised version (2005) is available online: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.695.8981

[6] Donald E. Simanek (2015), "Tidal Misconceptions." http://www.lhup.edu/ dsimanek/scenario/tides.htm. Accessed 21.12.2016.

[7] Paolo Sirtoli (2005), "Tides and centrifugal force." http://www.vialattea.net/maree/eng/index.htm. Accessed 21.12.2016.

[8] H. V. Thurman (1994), *Introductory Oceanography*, seventh edition. New York, NY: Macmillan pp. 252–276.